MATH416 COMPLEX ANALYSIS II

LECTURE NOTE

(3 CREDIT UNITS)

PROF. ALHAJI JIBRIL ALKALI(Ph.D)

DEPARTMENT OF MATHEMATICS
AHMADU BELLO UNIVERSITY, ZARIA,
NIGERIA

Course Outline

Taylor and Laurent series expansions. Isolated singularities and residues. Residue theorem Calculus of residue, and application to evaluation of integrals and to summation of series. Maximum Modulus principle. Argument principle. Rouche's theorem. The fundamental theorem of algebra. Principle of analytic continuation. Multiple valued functions and Riemann surfaces.

Contents

Chapter 1

Complex Power Series

1.1 Taylor's Series Representation

Definition 1.1.1 If $f(z)$ is analytic at $z = z_0$ then the series

$$\sum_{n=0}^{\infty} \frac{f^n(z_0)(z-z_0)^n}{n!} = f(z_0) + \frac{f'(z_0)(z-z_0)}{1!} + \frac{f''(z_0)(z-z_0)^2}{2!} + \cdots + \frac{f^{(n)}(z_0)(z-z_0)^n}{n!} + \cdots$$

is called the Taylor series expansion for $f(z)$ centred at $z = z_0$. When the centre is $z_0 = 0$, the series is called the Maclaurin;s series for $f(z)$ that is,

$$\sum_{n=0}^{\infty} \frac{f^n(z_0)z^n}{n!} = f(z_0) + \frac{f'(z_0)z)}{1!} + \frac{f''(z_0)z^2}{2!} + \cdots + \frac{f^{(n)}(z_0)z^n}{n!} + \cdots$$

Theorem 1.1.1 (**Taylor's Theorem**)
Suppose that $f(z)$ is analytic in the interior of a circle C with centre at Z_0 and radius R. Then the Taylor series for f converges to $f(z)$ for all z inside C. That is,

$$\begin{aligned} f(z) &= \textstyle\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!} \\ &= \textstyle\sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ where } a_n = \frac{f^{(n)}(z_0)}{n!} \end{aligned}$$

Proof.

Diagram

Let z be any point inside C. Construct a circle C_1 with centre at z_0 and radius r and enclosing z.
Let w be any point on $C_1 : |w - z_0| = r$.

Thus by Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w) \cdot dw}{w - z} \tag{1.1}$$

Consider

$$\begin{aligned}
\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} \\
&= \frac{1}{(w-z_0)\left[1-\dfrac{z-z_0}{w-z_0}\right]} \\
&= \frac{1}{w-z_0}\left[1-\frac{z-z_0}{w-z_0}\right]^{-1} \\
&= \frac{1}{w-z_0}\left\{1+\frac{z-z_0}{w-z_0}+\left(\frac{z-z_0}{w-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{w-z_0}\right)^{n-1}\right. \qquad (1.2) \\
&\quad \left.+\left(\frac{z-z_0}{w-z_0}\right)^n\cdot\left(\frac{1}{1-\dfrac{z-z_0}{w-z_0}}\right)\right\} \\
&= \frac{1}{w-z_0}+\frac{z-z_0}{(w-z_0)^2}+\frac{(z-z_0)^2}{(w-z_0)^3}+\cdots+\frac{(z-z_0)^{n-1}}{(w-z_0)^n} \\
&\quad +\left(\frac{z-z_0}{w-z_0}\right)^n\cdot\frac{1}{w-z}
\end{aligned}$$

$$\begin{aligned}\frac{1}{2\pi i}\int_{c_1}\frac{f(w)dw}{w-z} &= \frac{1}{2\pi i}\int_{c_1}\frac{f(w)dw}{w-z_0}+\frac{z-z_0}{2\pi i}\int_{c_1}\frac{f(w)dw}{(w-z_0)^2}+\cdots\\ &+\frac{(z-z_0)^{n-1}}{2\pi i}\int_{c_1}\frac{f(w)dw}{(w-z_0)^n}+\frac{1}{2\pi i}\int_{c_1}\frac{\left(\frac{z-z_0}{w-z_0}\right)^n f(w)dw}{w-z}\end{aligned} \tag{1.3}$$

Using Cauchy integral formula on (1.3) we have

$$f(z) = f(z_0)+\frac{(z-z_0)f'(z_0)}{1!}+\frac{(z-z_0)^2f''(z_0)}{2!}+\cdots+\frac{(z-z_0)^{n-1}}{(n-1)!}f^{(n-1)}(z_0)+R_n$$

$$\text{where } R_n = \frac{1}{2\pi i}\oint_{c_1}\frac{\left(\frac{z-z_0}{w-z_0}\right)^n f(w)dw}{w-z} \tag{1.4}$$

To prove the required result, we need to show that

$$\lim_{n\to\infty}|R_n|=0$$

since w is on the C_1 then $\left|\frac{z-z_0}{w-z_0}\right| = k < 1$.

Since $f(w)$ is analytic, then $|f(w)| \le M$, where M is a constant.

$$|w-z| = |(w-z_0)-(z-z_0)| \ge |w-z_0|-|z-z_0| = r-|z-z_0|$$

Hence

$$\begin{aligned}|R_n| &= \left|\frac{1}{2\pi i}\int_{C_1}\left(\frac{z-z_0}{w-z_0}\right)^n\frac{f(w)dw}{w-z}\right|\\ &\le \frac{1}{2\pi}\int_{C_1}\left|\frac{z-z_0}{w-z_0}\right|^n\frac{|f(w)|}{|w-z|}|dw|\\ &< \frac{1}{2\pi}\int_{C_1}\frac{k^nM}{r-|z-z_0|}|dw|\\ &< \frac{1}{2\pi}\cdot\frac{k^nM}{r-|z-z_0|}\int_{C_1}|dw|\\ &< \frac{1}{2\pi}\cdot\frac{k^nM}{r-|z-z_0|}2\pi r = \frac{k^nMr}{r-|z-z_0|}\end{aligned}$$

$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \frac{k^n Mr}{r - |z - z_0|} = 0$$

That is, $\lim_{n\to\infty} |R_n| = 0$ hence, the result.

$\square$

Note: If a function f is analytic at a point z_0, then it can be expanded in a convergent power series about that point. If $z - z_0 = h$, then $z = z_0 + h$ and

$$f(z_0 + h) = f(z_0) + \frac{h}{1!} f'(z_0) + \frac{h^2}{2!} f^{(2)}(z_0) + \cdots + \frac{h^n}{n!} f^{(n)}(z_0) = \sum_{n=0}^{\infty} \frac{h^n f^{(n)}(z_0)}{n!}$$

Example 1.1.1 Expand $\frac{1}{z^2}$ in the region $|z - 2| < 2$.

Solution:

Example 1.1.2 Expand $\frac{1}{5z + 1}$ in Taylor's series about the point $z = 3$

Solution:

Example 1.1.3 Expand $f(z) = \ln(1 + z)$ about the point $z = 0$

Solution:

Example 1.1.4 Expand $\sin 2z$ about the point $z = \frac{\pi}{2}$.

Solution:

Exercise 1

1. Expand each given function in a Taylor series expansion about the indicated points

   a) $\ln\left(\frac{1 + z}{1 - z}\right)$ about $z = 0$.

b) $\log z = \log|z| + iarg(z)$ about $z = 2$.

c) $\dfrac{1}{2z-3}$ about $z = 2$.

d) $\dfrac{z+i}{z-i}$ about $z = 1+1$

e) e^{z-1} about $z = 2$.

f) $\dfrac{2z}{z-2}$ about $z = 1$

2. In each of the following functions, expand them at the indicated points and the region of convergence.

a) $\dfrac{\sin z}{z^2+4}$, $z = 0$

b) $\dfrac{z+3}{(z-1)(z-4)}$, $z = 2$

c) $\dfrac{e^z}{z(z-1)}$, $z = 4i$

3. Show that

a) $e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots + \dfrac{z^n}{n!}$, $|z| < \infty$

b) $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots + \dfrac{(-1)^n z^{2n-1}}{(2n-1)!}$, $|z| < \infty$

c) $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \cdots + \dfrac{(-1)^{n-1} z^{2n-2}}{(2n-2)!}$, $|z| < \infty$.

d) $\tan^{-1} z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots + \dfrac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$, $|z| < \infty$

1.2 Laurent Series Representation

Suppose that $f(z)$ is not analytic in a circle C with centre at $z = z_0$ and radius R, but is analytic in the punctured region $D = \{z : 0 < |z - z_0| < R\}$. For example, the function $f(z) = \dfrac{e^z}{z^3}$, is not analytic at $z = 0$ but is analytic for $|z| > 0$. Clearly, the function does not have Maclaurin Series representation.

However,

$$\begin{aligned} f(z) = \frac{e^z}{z^3} &= \frac{1}{z^3}\left\{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right\} \\ &= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!z} + \frac{1}{3!} + \frac{z}{4!} + \cdots \\ &= z^{-3} + z^{-2} + \frac{1}{2!}z^{-1} + \frac{z}{4!} + \cdots \end{aligned}$$

which is valid for all z such that $|z| > 0$. we may then represent this type of function with a series that involve positive and negative power of z.

1.2.1 Laurent Series

Let a_n be a complex number for $n = 0, \pm 1, \pm 2, \cdots$ The infinite series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, is called a Laurent series, is expanded as

$$\begin{aligned} \textstyle\sum_{-\infty}^{\infty} a_n(z-z_0)^n &= \textstyle\sum_{n=-\infty}^{-1} a_n(z-z_0)^n + \sum_{n=0}^{\infty} a_n(z-z_0)^n \\ &= \textstyle\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n \end{aligned}$$

Provided the series in the R.H.S of this equation converges.

Theorem 1.2.1 Laurent Theorem
If $f(z)$ is analytic in the annulus (ring shaped region) between two concentric circles C_1 and C_2 with centre at $z = z_0$ and radii r_1 and $r_2 (r_2 > r_1)$, then at any point z within the annulus

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

where

$$a_n = \frac{1}{2\pi}\int_{C_2} \frac{f(w)dw}{(w-z_0)^{n+1}},\ n = 0, 1, 2, \cdots$$

$$a_{-n} = \frac{1}{2\pi}\int_{C_1} \frac{f(w)dw}{(w-z_0)^{-n+1}},\ n = 1, 2, 3, \cdots$$

Proof. Construct two concentric circles C_1 and C_2 as seen in the diagram below

Diagram

The integral around C_2 and C_1 being taken in the positive and negative directions respectively. By Cauchy integral formula for multiply connected region

$$f(z) = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)dw}{(w-z)} - \frac{1}{2\pi i}\int_{C_1}\frac{f(w)dw}{(w-z)} \tag{1.5}$$

Case 1: Consider the integral $\frac{1}{2\pi i}\int_{C_2}\frac{f(w)dw}{w-z}$

$$\begin{aligned}
\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} \\
&= \frac{1}{(w-z_0)\left[1-\dfrac{z-z_0}{w-z_0}\right]} \\
&= \frac{1}{w-z_0}\left[1-\frac{z-z_0}{w-z_0}\right]^{-1} \\
&= \frac{1}{w-z_0}\left\{1+\frac{z-z_0}{w-z_0}+\left(\frac{z-z_0}{w-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{w-z_0}\right)^{n-1}\right. \\
&\quad \left. +\left(\frac{z-z_0}{w-z_0}\right)^n\cdot\left(\frac{1}{1-\dfrac{z-z_0}{w-z_0}}\right)\right\} \\
&= \frac{1}{w-z_0}+\frac{z-z_0}{(w-z_0)^2}+\frac{(z-z_0)^2}{(w-z_0)^3}+\cdots+\frac{(z-z_0)^{n-1}}{(w-z_0)^n} \\
&\quad +\left(\frac{z-z_0}{w-z_0}\right)^n\cdot\frac{1}{w-z}
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{c_2}\frac{f(w)dw}{w-z} &= \frac{1}{2\pi i}\int_{c_2}\frac{f(w)dw}{w-z_0} + \frac{z-z_0}{2\pi i}\int_{c_2}\frac{f(w)dw}{(w-z_0)^2} + \cdots \\
&\quad + \frac{(z-z_0)^{n-1}}{2\pi i}\int_{c_2}\frac{f(w)dw}{(w-z_0)^n} + \frac{1}{2\pi i}\int_{c_2}\frac{\left(\frac{z-z_0}{w-z_0}\right)^n f(w)dw}{w-z} \\
&= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + R_n
\end{aligned} \tag{1.6}
$$

$$
\text{where } R_n = \frac{1}{2\pi i}\int_{c_2}\frac{\left(\frac{z-z_0}{w-z_0}\right)^n f(w)dw}{w-z}
$$

Consider the integral $\frac{1}{2\pi i}\int_{C_1}\frac{f(w)dw}{w-z}$ from (1.5);

$$
\begin{aligned}
\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} \\
&= \frac{1}{(w-z_0)\left[1-\frac{z-z_0}{w-z_0}\right]} \\
&= \frac{1}{w-z_0}\left[1-\frac{z-z_0}{w-z_0}\right]^{-1} \\
&= \frac{1}{w-z_0}\left\{1+\frac{z-z_0}{w-z_0}+\left(\frac{z-z_0}{w-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{w-z_0}\right)^{n-1}\right. \\
&\quad \left. +\left(\frac{z-z_0}{w-z_0}\right)^n\cdot\left(\frac{1}{1-\frac{z-z_0}{w-z_0}}\right)\right\} \\
&= \frac{1}{w-z_0}+\frac{z-z_0}{(w-z_0)^2}+\frac{(z-z_0)^2}{(w-z_0)^3}+\cdots+\frac{(z-z_0)^{n-1}}{(w-z_0)^n} \\
&\quad +\left(\frac{z-z_0}{w-z_0}\right)^n\cdot\frac{1}{w-z}
\end{aligned}
$$

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c_1}\frac{f(w)dw}{w-z} &= \frac{1}{2\pi i}\int_{c_1}\frac{f(w)dw}{w-z_0}+\frac{z-z_0}{2\pi i}\int_{c_1}\frac{f(w)dw}{(w-z_0)^2}+\cdots \\
&+\frac{(z-z_0)^{n-1}}{2\pi i}\int_{c_1}\frac{f(w)dw}{(w-z_0)^n}+\frac{1}{2\pi i}\int_{c_1}\frac{\left(\dfrac{z-z_0}{w-z_0}\right)^n f(w)dw}{w-z} \\
&= \frac{a_{-1}}{z-z_0}+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-3}}{(z-z_0)^3}+\cdots+\frac{a_{-n}}{(z-z_0)^n}+T_n
\end{aligned} \tag{1.7}$$

$$\text{where } T_n = \frac{1}{2\pi i}\int_{c_1}\frac{\left(\frac{z-z_0}{w-z_0}\right)^n f(w)dw}{w-z}$$

Combing (1.5), (1.6), and (1.7) yield:

$$\begin{aligned}
f(z) &= \{a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots+a_{n-1}(z-z_0)^{n-1}\} \\
&+\frac{a_{-1}}{z-z_0}+\frac{a_{-2}}{(z-z_0)^2}+\cdots+\frac{a_{-n}}{(z-z_0)^n}+R_n+T_n
\end{aligned} \tag{1.8}$$

To finally prove the theorem, we need to show that

(i) $|R_n| \to 0$ as $n \to \infty$

(ii) $|T_n| \to 0$ as $n \to \infty$

(i) Since w is on the C_2 then $\left|\dfrac{z-z_0}{w-z_0}\right| = k < 1$.

Since $f(w)$ is analytic, then $|f(w)| \le M$, where M is a constant.

$$|w-z| = |(w-z_0)-(z-z_0)| \ge |w-z_0|-|z-z_0| = r_2-|z-z_0|$$

Hence

$$\begin{aligned}
|R_n| &= \left|\frac{1}{2\pi i}\int_{C_2}\left(\frac{z-z_0}{w-z_0}\right)^n\frac{f(w)dw}{w-z}\right| \\
&\le \frac{1}{2\pi}\int_{C_2}\left|\frac{z-z_0}{w-z_0}\right|^n\frac{|f(w)|}{|w-z|}|dw| \\
&< \frac{1}{2\pi}\int_{C_2}\frac{k^nM}{r_2-|z-z_0|}|dw| \\
&< \frac{1}{2\pi}\cdot\frac{k^nM}{r_2-|z-z_0|}\int_{C_2}|dw| \\
&< \frac{1}{2\pi}\cdot\frac{k^nM}{r_2-|z-z_0|}2\pi r_2 = \frac{k^nMr_2}{r_2-|z-z_0|} \\
&\lim_{n\to\infty}|R_n| = \lim_{n\to\infty}\frac{k^nMr_2}{r_2-|z-z_0|} = 0
\end{aligned}$$

That is, $\lim_{n\to\infty}|R_n| = 0$

(ii) Since w is on the C_1 then $\left|\frac{z-z_0}{w-z_0}\right| = k < 1, k = \text{constant}$

Since $f(w)$ is analytic, then $|f(w)| \le M$, where M is a constant, and

$$|z-w| = |(z-z_0)-(w-z_0)| \ge |z-z_0| - |w-z_0| \ge |z-z_0| - r_1$$

Hence

$$\begin{aligned}
|T_n| &= \left|\frac{1}{2\pi i}\int_{C_1}\left(\frac{w-z_0}{z-z_0}\right)^n\frac{f(w)dw}{z-w}\right| \\
&\le \frac{1}{2\pi}\int_{C_1}\left|\frac{w-z_0}{z-z_0}\right|^n\frac{|f(w)|}{|z-w|}|dw| \\
&\le \frac{1}{2\pi}\int_{C_1}\frac{k^nM}{|z-z_0|-r_1}|dw| \\
&< \frac{1}{2\pi}\cdot\frac{k^nM}{|z-z_0|-r_1}\int_{C_2}|dw| \\
&< \frac{1}{2\pi}\cdot\frac{k^nM}{|z-z_0|-r_1}2\pi r_1 = \frac{k^nMr_1}{|z-z_0|-r_1}
\end{aligned}$$

$$\lim_{n\to\infty} |T_n| = \lim_{n\to\infty} \frac{k^n M r_1}{|z - z_0| - r_1} = 0$$

That is, $\lim_{n\to\infty} |T_n| = 0$

$$\therefore\ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

□

Example 1.2.1 1. Find the Laurent series about the indicated singularity for each of the following functions and give the region of convergence of each series:

(i) $\dfrac{e^{2z}}{(z-1)^3};\ z = 1,$

(ii) $(z-3)\dfrac{1}{\sin(z+2)};\ z = -2,$

(iii) $\dfrac{z - \sin z}{z^3};\ z = 0,$

(iv) $\dfrac{z}{(z+1)(z+2)};\ z = -2,$

(v) $\dfrac{1}{z^2(z-3)^2};\ z = 3$

Solution:

(i)

(iii)

(iv)

(ii) and (v) Exercise.

2. Find the Laurent series expansion in power of $z+1$ for the function defined by

$$f(z) = \frac{z}{(z-1)(z-2)}$$

in the region;

(i) $0 < |z+1| < 2$

(ii) $2 < |z+1| < 3$

(iii) $|z+1| > 3$

Solution

(i)

(ii)

(iii)

Exercise 2

1. Show that Laurent series expansion in powers of $z+1$, which represent the function f defined by $f(z) = \dfrac{z^2+1}{z(z^2+3z+2)}$ in the region $|z+1| > 3$ is given by
$$\frac{1}{2}\sum_{n=0}^{\infty}(1 - 2^{n+2} + 5 \cdot 3^n)(z+1)^{-n+1}.$$

2. Expand $f(z) = \dfrac{1}{z^2+1}$, in powers of $z+i$ in the regions

   (i) $0 < |z+1| < 2$

   (ii) $|z+1| > 2$

3. Expand in Laurent series in powers of $z-1$ which represents the function $f(z) = \dfrac{z}{(z-1)(z+1)(z+2)}$ in the regions

   (i) $0 < |z-1| < 2$

   (ii) $2 < |z-1| < 3$

   (iii) $|z-1| > 3$

1.3 Singular Points

Definition 1.3.1 A point at which the function $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$.

Definition 1.3.2 The point z_0 is called an isolated singularity of $f(z)$ if we can find some neighbourhood of z_0 that encloses no any other singular point other than z_0.

Example 1.3.1

1. The function f defined by $f(z) = \dfrac{1}{z-i}$ is analytic $\forall z \in \mathbb{C}$ except at the point $z = i$. Thus, $z = i$ is an isolated singular point of $f(z)$.

2. $f(z) = \dfrac{1}{z}$ has an isolated singularity at $z = 0$, because the region $|z| = r$ contains no singular point , other than $z = 0$ within it.

3. The function $f(z) = \dfrac{z-1}{z(z^2+1)}$ has three isolated singularities, $z = 0, -i, i$.

4. Branch point of multivalued functions are singular point such as

   (i) $f(z) = (z-5)^{\frac{1}{2}}$ has branch point(s) where $(z-5)^{\frac{1}{2}} = 0$, that is, $z = 5$.

   (ii) $f(z) = \ln(z^2 + z - 2)$ has branch points where $z^2 + z - 2 = 0$, that is, $z = -2, 1$.

Definition 1.3.3 A singularity which is not isolated is called non-isolated singularity. For example $f(z) = e^{\frac{1}{z-2}}$ has non-isolated (or essential) singularity at $z = 2$.

Suppose that z_0 is an isolated singular point of a function f and f is analytic in the annulus $D : 0 < |z - z_0| < r$, where r is the distance from z_0 to the nearest singular point of f other than z_0 itself.
The utilising Laurent series we have $\forall z \in D$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(w)dw}{(w - z_0)^{n+1}}, n = 0, \pm 1, \pm 2, \cdots$$

That is,

$$f(z) = \sum_{n=-1}^{-\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1.9}$$

the part $\sum_{n=-1}^{-\infty} a_n(z-z_0)^n$ is called the principal part while $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is called the analytic part.

It is the principal part that reveals the character or nature of the singularity of f at z_0.

Three types of singularity are distinguished according to the following:

Type I: Suppose all coefficients in the principal part are zero, that is, $a_n = 0, n = 1, 2, 3, \cdots$, then we may write (1.9)as

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, z \neq z_0.$$

If

$$f(z_0) = a_0 = \lim_{z\to z_0} f(z)$$

then a singularity of this type is said to be a removable singularity.

For example $f(z) = \dfrac{\sin z}{z}$ has singularity at $z = 0$.

$$\begin{aligned} \therefore\ f(z) = \frac{\sin z}{z} &= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots \end{aligned}$$

Thus $f(0) = 1$ and $\lim_{z\to 0} f(z) = \lim_{z\to 0} \dfrac{\sin z}{z} = 1$. Hence, $f(z)$ has a removable singularity at $z = 0$.

Type II: Suppose that the principal part has a finite number of terms that is,

$$\begin{aligned} f(z) &= a_{-m}(z-z_0)^{-m} + a_{-m+1}(z-z_0)^{-m+1} + \cdots + a_{-2}(z-z_0)^2 \\ &+ a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \\ &= \textstyle\sum_{n=-1}^{-m}(z-z_0)^n + \sum_{n=0}^{\infty},\ a_{-m} \neq 0 \end{aligned}$$

In this case, $z = z_0$ is said to be a Pole of order (or multiplicity)m. In particular, when $m = 1, z = z_0$ is said to be a simple pole.

For example $f(z) = \dfrac{\sin z}{z^4}$ has singularity at $z = 0$.

$$\begin{aligned} \therefore\ f(z) = \frac{\sin z}{z^4} &= \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) \\ &= \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots \end{aligned}$$

Hence, $z = 0$ is a pole of order 3.

Type III: Suppose the principal part has an infinite number of terms, the point $z = z_0$ is said to be an essential singularity of f.
For example

$$e^{\frac{1}{z}} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots + \frac{1}{n!z^n} + \cdots$$

Exercise 3

1. Find the Laurent series about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of convergence of each series

   (i) $f(z) = \dfrac{e^z}{(z-1)^2}$; $z = 1$.

   (ii) $f(z) = z \cos \dfrac{1}{z}$; $z = 0$.

   (iii) $f(z) = \dfrac{\sin z}{z - \pi}$; $z = \pi$.

   (iv) $f(z) = \dfrac{z}{(z+1)(z+2)}$; $z = -1$.

   (v) $f(z) = \dfrac{1}{z(z+2)^3}$; $z = 0$.

   (vi) $f(z) = \dfrac{1}{z(z+3)^3}$; $z = -2$.

2. find the Laurent series about the singular point of the given function and name the singularity

   (i) $f(z) = z^3 e^{\frac{1}{z}}$.

   (ii) $f(z) = \dfrac{z - \sin z}{z^3}$.

   (iii) $f(z) = \dfrac{1 - \cos z}{z^5}$.

   (iv) $f(z) = e^{\frac{1}{(z-1)^2}}$.

   (v) $f(z) = \dfrac{1 - \cos z}{z}$.

3. Expand $e^{z^2} + e^{\frac{1}{z^2}}$ in Laurent series valid for $|z| > 0$

4. Find the principal parts of the following Laurent series

   (i) $\dfrac{z^2}{z^4 - 1}$.

   (ii) $\dfrac{e^z}{z^4}$ for $|z| > 0$

1.4 Calculus of Residue

Let $f(z)$ be analytic everywhere inside and on a simple closed contour C except at the point $z = z_0$ which is a pole of order m. Then $f(z)$ can be expanded as a Laurent series at $z = z_0$ as

$$\begin{aligned} f(z) &= \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 \\ &+ a_1(z-z_0) + \frac{a_2(z-z_0)^2}{2!} + \cdots + \frac{a_n(z-z_0)^n}{n!} + \cdots \quad a_n \neq 0 \end{aligned} \tag{1.10}$$

Integrating (1.10) along C positively, we have

$$\begin{aligned} \int_c f(z)dz &= \int_C \frac{a_{-m}}{(z-z_0)^m}dz + \int_C \frac{a_{-m+1}}{(z-z_0)^{m-1}}dz + \cdots + \\ & \int_C \frac{a_{-1}}{(z-z_0)}dz + \int_c \sum_{n=0}^{\infty} a_n(z-z_0)^n dz \\ &= 0 + 0 + \cdots + a_{-1}(2\pi i) + 0 + 0 + \cdots + 0 \\ &= 2\pi i(a_{-1}) \end{aligned}$$

that is,

$$a_{-1} = \frac{1}{2\pi i}\int_c f(z)dz$$

a_{-1} is called the residue of $f(z)$ at the point $z = z_0$.
We may also denote the residue of $f(z)$ at the pole $z = z_0$ by $Res(f(z), z_0)$ or $Res(z_0)$.

Example 1.4.1 If $f(z) = \dfrac{1}{z^5} + \dfrac{2}{z^4} + \dfrac{2}{z^3} + \dfrac{2^2}{3z^2} + \dfrac{2^4}{4!z} + \dfrac{2^5}{5!} + \dfrac{2^6 z}{6!} + \cdots$ then $a_{-1} = \dfrac{2^4}{4!} = \dfrac{2}{3}$ that is $Res(f(z), z_0) = Res(0) = \dfrac{2}{3}$.

We can also use formula to find the residue of a function at a given pole.

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots \quad (1.11)$$

then by multiplying (1.11) by $(z-z_0)^n$ gives Taylor's series

$$(z-z_0)^n f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{n-1} + a_0(z-z_0)^n + a_1(z-z_0)^{n+1} + \cdots$$

$$\begin{aligned}
\frac{d^{n-1}}{dz^{n-1}}\{(z-z_0)^n f(z)\} &= \frac{d^{n-1}}{dz^{n-1}}\{a_{-n} + a_{-n+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{n-1} \\
&\quad + a_0(z-z_0)^n + a_1(z-z_0)^{n+1} + \cdots\} \\
&= 0 + 0 + (n-1)! a_{-1} \\
&\quad + \frac{d^{n-1}}{dz^{n-1}}\{a_0(z-z_0)^n + a_1(z-z_0)^{n+1} + \cdots\}
\end{aligned}$$

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]$$

That is,

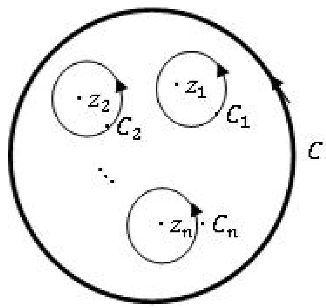
$$Res(f(z), z_0) = Res(z_0) = \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]$$

Theorem 1.4.1 Suppose $f(z)$ is analytic inside and on the simple closed contour C except for isolated singularities at $z_1, z_2, z_3, \cdots, z_n$ inside C with residues $Res(z_1, Res(z_2, Res(z_3, \cdots, Res(z_n)$ respectively. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res(z_k).$$

Proof.

About each singularity z_i, construct a circle C_i in C and such that $C_i \cap C_j = \phi$ for $i \neq j$. Then by Cauchy integral theorem for multiply connected region



$$\begin{aligned}\int_C f(z)dz &= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots = \int_{C_n} f(z)dz \\ &= 2\pi i Res(z_1) + 2\pi i Res(z_2) + \cdots + 2\pi i Res(z_n) \\ &= 2\pi i\{Res(z_1) + Res(z_2) + \cdots + Res(z_n)\} \\ &= 2\pi i \sum_{k=1}^{n} Res(z_k).\end{aligned}$$

□

Example 1.4.2 Determine the order of each pole and the value of the residue of the function defined by

$$f(z) = \frac{e^z}{z^4 + z^2}.$$

Hence

$$\int_C \frac{e^z}{z^4 + z^2} dz,\ C : |z| = \frac{3}{2}.$$

Solution:

Example 1.4.3 Find the residue of $e^{zt} \tan z$ at a simple pole $z = \frac{3\pi}{2}$.

Solution:

Example 1.4.4 Show that

$$\int_C \frac{\sin z}{z^4} dz = -\frac{\pi i}{3},$$

where C is the circle described in the positive direction.

Solution:

Exercise 4

1. Evaluate the residue of $\dfrac{e^{az}}{\sinh \pi z}$, at each pole for $a > 0$.
2. Find the residue of $f(z) = \tan z$.
3. Find the residue of $f(z) = \dfrac{\cot z \coth z}{z^3}$ at $z = 0$.
4. Show that
$$\int_C \frac{e^z}{\sinh z} dz = 6\pi i,\ C : |z| = 4.$$
5. Show that
$$\int_C \left(\frac{1+z^5}{z^6}\right) \sinh z dz = \frac{11i}{60}.$$
6. Evaluate
$$\int_C e^{\frac{1}{z}} dz$$
7. Evaluate
$$\int_C \frac{z^2}{z^4+1} dz, C : |z| = 6.$$
8. Evaluate
$$\int_C \frac{z+4}{z^2-3z-10} dz,$$
along the circle (i) $C : |z| = 4$ (ii) $C : |z| = 6$.
9. Find the residues of (a) $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ (b) $f(z) = e^z \cos ecz$, all its poles inside finite plane.
10. Evaluate
$$\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2(z^2+2z+2)} dz, C : |z| = 3.$$

1.5 Integrals Involving Circular Function

We are familiar with the idea that the unit circle $C : |z| = 1$ has the equation $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$ that is, $z = \cos\theta + i\sin\theta$ and $z^{-1} = \cos\theta - i\sin\theta$. So that $\cos\theta = \dfrac{z+z^{-1}}{2}, \sin\theta = \dfrac{z-z^{-1}}{2i}$ and $dz = ie^{i\theta}d\theta = izd\theta$ or $d\theta = \dfrac{dz}{iz}$ we then deduce that

$$\begin{aligned}
\int_{|z|=1} f(z)dz &= \int_0^{2\pi} f\left(e^{i\theta}\right) ie^{i\theta}d\theta \\
&= \int_0^{2\pi} f\left(\cos\theta + i\sin\theta\right) ie^{i\theta}d\theta \\
&= \int_0^{2\pi} f\left(\cos\theta, \sin\theta\right) ie^{i\theta}d\theta \\
&= \int_0^{2\pi} f\left(\cos\theta, \sin\theta\right) izd\theta \\
\therefore \int_0^{2\pi} f\left(\cos\theta, \sin\theta\right) d\theta &= \frac{-i}{z}\int_{|z|=1} f(z)dz
\end{aligned}$$

It follows that if we are given an integral of the form

$$I = \int_0^{2\pi} F\left(\cos\theta, \sin\theta\right) d\theta \tag{1.12}$$

Where $F(\cos\theta, \sin\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$, which is finite over the range of integration. Then we consider the substitution $z = e^{i\theta}, \cos\theta = \dfrac{z+z^{-1}}{2}, \sin\theta = \dfrac{z-z^{-1}}{2i}$ and $d\theta = \dfrac{dz}{iz}$ so that (1.12) will be transformed into the integral

$$I = \int_C f(z)dz,$$

where $f(z)$ is finite over the path of integration $C : |z| = 1$.

Example 1.5.1 Evaluate

$$\int_0^{2\pi} \frac{d\theta}{1+3\cos^2\theta}$$

Solution:

Example 1.5.2 Prove that

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)d\theta = \frac{2\pi}{n!}, n = 0, 1, 2, \cdots$$

Solution:

Exercise 5

1. Show that
$$\int_0^{2\pi} \frac{1+\cos\theta}{2+\cos\theta}d\theta = \frac{2\pi}{\sqrt{3}}(\sqrt{3}-1)$$

2. Show that
$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}, \{a > b,\ b \neq 0$$

3. Show that
$$\int_0^{2\pi} e^{-\cos\theta}\{\cos(n\theta + \sin\theta)\}d\theta = (-1)^n\frac{2\pi}{n!}$$

4. Show that
$$\int_0^{2\pi} \frac{\cos n\theta}{1-2a\cos\theta + a^2}d\theta = \frac{2\pi a^n}{1-a^2}, n = 0, 1, 2, \cdots$$

5. Show that
$$\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}d\theta = \frac{-\pi}{3}$$

6. Show that
$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin^2\theta} = \frac{\pi}{2\sqrt{2}}$$

Review of Some Complex Concepts

1. Recall that for any integer $n = 0, \pm1, \pm2, \pm3, \cdots$ and a circle C with centre at z_0 and radius r positively oriented, then
$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1 \end{cases}$$

Or

$$\oint_C (z-z_0)^n dz = \begin{cases} 2\pi i & n=-1 \\ \\ 0 & n \neq -1 \end{cases}$$

2. **Cauchy Integral Theorem :**
   Let a complex function $f(z)$ be analytic in a simply connected region D and C be a simple closed contour that lies in D. then

$$\oint_C f(z)dz = 0$$

3. **Cauchy Integral Formula:**
   Let a complex function say $f(z)$ be analytic inside and on a simple closed curve C and z_0 is any point in C. Then $f^{(n)}(z_0)$ exists for $n = 0, 1, 2, 3, \cdots$ and is given by

   (i)
$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}$$

   and in particular when $n = 0$

   (ii)
$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)}$$

Some Consequences of Cauchy Integral Formula

1. **Morera's Theorem**

   Theorem 1.5.1 Morera's Theorem
   Let f be a continuous function in a simply connected domain D and $\int_C f(z)dz = 0$ for every closed contour C in D, then f is analytic in D.

   Proof. We first prove that $\int_C f(z)dz$ is independent of the path joining any two points z_0 and z in D

   Diagram here

We let C_1 and C_2 be two contour in D, both with initial point z_0 and terminal point z, as shown in the diagram above.
$\therefore\ C = C_1 \cup C_2$ is now a simple closed contour and so

$$\int_C f(z)dz = \int_{C_1 \cup C_2} f(z)dz = 0\,(hypothesis)$$

that is,

$$\int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0$$

that is,

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

that is,

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz = \int_{z_0}^{z} f(z)dz$$

We now show that $f(z)$ is analytic.
Let z_0 be held fixed and let $|h|$ be chosen small enough so that the point $z + h$ lies in the domain D.

Diagram here

With z_0 in D, we define F by

$$F(z) = \int_{z_0}^{z} f(u)du, \forall u, z \in D$$

$$\begin{aligned} F(z+h) - F(z) &= \textstyle\int_{z_0}^{z+h} f(u)du - \int_{z_0}^{z} f(u)du \\ &= \textstyle\int_{z_0}^{z} f(u)du + \int_{z}^{z+h} f(u)du - \int_{z_0}^{z} f(u)du \\ &= \textstyle\int_{z}^{z+h} f(u)du \end{aligned}$$

and

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_{z}^{z+h} f(u)du$$

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_{z}^{z+h} f(u)du - f(z)$$

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_{z}^{z+h} f(u)du - \frac{f(z)}{h}\int_{z}^{z+h} du$$

$$\frac{F(z+h)-F(z)}{h}-f(z)=\frac{1}{h}\int_z^{z+h}\{f(u)-f(z)\}du \tag{1.13}$$

Since f is continuous in D, then for every $\epsilon > 0 \exists \delta > 0 \ni \ |f(u)-f(z)| < \epsilon$ whenever $|z-u| < \delta$

$$\begin{aligned}
\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| &= \left|\frac{1}{h}\int_z^{z+h}\{f(u)-f(z)\}du\right| \\
&\leq \frac{1}{|h|}|f(u)-f(z)|\int_z^{z+h}|du| \\
&< \frac{1}{|h|}\cdot\epsilon\int_z^{z+h}|du| \\
&< \frac{1}{|h|}\cdot\epsilon\cdot|h| \\
&< \epsilon
\end{aligned}$$

that

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| < \epsilon \text{ providee } |h| < \delta.$$

That is,

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| \to 0 \text{ as } |h| \to 0.$$

That is

$$\lim_{|h|\to 0}\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| = 0.$$

That is,

$$\lim_{|h|\to 0}\frac{F(z+h)-F(z)}{h}-\lim_{|h|\to 0}f(z) = 0.$$

That is,

$$\lim_{|h|\to 0}\frac{F(z+h)-F(z)}{h}-f(z) = 0.$$

That is,

$$\lim_{|h|\to 0}\frac{F(z+h)-F(z)}{h} = f(z).$$

that is,

$$F'(z) = f(z)$$

exists.
Hence, f being derivative of analytic function $F(z)$, is also analytic. $\square$

2. **Cauchy Inequality**

Theorem 1.5.2 Cauchy Inequality
If $f(z)$ is analytic inside and on a circle C of radius r and centre at $z = z_0$, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{Mn!}{r^n}, n = 0, 1, 2, \cdots$$

where M is a constant $\ni |f(z)| \leq M$

Proof. This theorem has all the properties of Cauchy integral formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz, n = 0, 1, 2, \cdots \tag{1.14}$$

$$\begin{aligned}
|f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \\
&\leq \left| \frac{n!}{2\pi i} \right| \oint_C \left| \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \\
&\leq \frac{n!}{2\pi} \oint_C \frac{|f(z)|}{|z - z_0|^{n+1}} |dz| \\
&\leq \frac{n!}{2\pi} \oint_C \frac{M}{r^{n+1}} |dz|
\end{aligned}$$

$$|f^{(n)}(z_0)| \leq \frac{Mn!}{2\pi r^{n+1}} \oint_C |dz| \tag{1.15}$$

the equation of C is $C : z - z_0 = re^{i\theta}, 0 \leq \theta \leq 2\pi$
that is, $z = z_0 + re^{i\theta}$ and $dz = ire^{i\theta} d\theta$
so that $|dz| = |ire^{i\theta} d\theta| = r d\theta$
thus, (1.15) reduce to

$$|f^{(n)}(z_0)| \leq \frac{Mn!}{2\pi r^{n+1}} \oint_0^{2\pi} r d\theta = \frac{Mn!}{r^n}, n = 0, 1, 2, 3, \cdots$$

$\square$

Theorem 1.5.3 If the function $f(z)$ is analytic and bounded for all values of z in the complex plane $\mathbb{C}$, then $f(z)$ is constant.

Proof. given $f(z)$ is bounded for all $z \in \mathbb{C}$ implies there exists a constant number $M > 0$ such that $|f(z)| \leq M$.

Diagram here

Let z_0 be any point in $\mathbb{C}$ and let C be a circle centre at z_0 and radius r.
Since $f(z)$ is analytic for all $z \in \mathbb{C}$, then by Cauchy inequality with $n = 1$, we have

$$|f'(z_0)| \leq \frac{M}{r}.$$

Letting $r \to \infty$, we deduce that $|f'(z_0)| = 0 \Longleftrightarrow f'(z_0) = 0$. Since z_0 is any point in $\mathbb{C}$, z_0 can be replaced by z so that $f'(z) = 0 \Longleftrightarrow f(z) =$constant. □

Theorem 1.5.4 (Fundamental Theorem of Algebra)
Every polynomial equation

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$$

of degree $n \geq 1$ and $a_n \neq 0$ has atleast one root.

Proof. Supposed that no value of z exists for which $p(z) = 0$. That is, $p(z) \neq 0 \forall z \in \mathbb{C}$. Thus $f(z) = \dfrac{1}{p(z)}$ is analytic for all $z \in \mathbb{C}$. As $r = |z| \to \infty$ then

$$|f(z)| = \frac{1}{|p(z)|} \to 0$$

implying that $f(z) = \dfrac{1}{p(z)}$ is bounded. and so $p(z) = \dfrac{1}{f(z)}$ is also bounded.
Hence, by Liouville's theorem $p(z)$ is constant. this contradicts the hypothesis that $p(z)$ is of degree $n \geq 1$.
Therefore, our assumption that $p(z) \neq 0$ is not valid, hence the result. □

Corollary 1.5.5 Every polynomial equation

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$$

where the degree $n \geq 1$ and $a_n \neq 0$ has n roots.

Proof. By the fundamental theorem of algebra, $p(z)$ has atleast one root. Denote this root by α_1. Then $p(\alpha_1) = 0$. Hence,

$$\begin{aligned} p(z) - P(\alpha_1) &= (a_0 + a_1 z + \cdots + a_n z^n) - (a_0 + a_1\alpha_1 + \cdots + a_n\alpha_1^n) \\ &= a_1(z - \alpha_1) + a_2(z^2 - \alpha_1^2) + \cdots + a_n(z^n - \alpha_1^n) \\ &= (z - \alpha_1)\{a_1 + a_2(z - \alpha_1) + \cdots\} \\ &= (z - \alpha_1)Q_1(z) \end{aligned}$$

That is,

$$p(z) - p(\alpha_1) = (z - \alpha_1)Q_1(z) \tag{1.16}$$

where $Q_1(z)$ is a polynomial of degree $(n - 1)$. Applying the fundamental theorem of Algebra again on $Q_1(z)$, $Q_1(z)$ will have at least one root say α_2(which may be equal to α_1) and so (1.16) becomes

$$\begin{aligned} p(z) - P(\alpha_1) &= (z - \alpha_1)(z - \alpha_2)Q_2(z) \\ &= (z - \alpha_1)(z - \alpha_2)(z - \alpha_3)Q_3(z) \\ &\vdots \\ &= (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n) \end{aligned}$$

That is

$$p(z) - 0 = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)$$

That is

$$p(z) = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)$$

Showing that $p(z)$ has exactly n roots. □

Theorem 1.5.6 Maximum Modulus Theorem
If f is a non-constant analytic function inside and on a simply closed contour C, then the maximum and minimum value of $|f(z)|$ occurs on C.

Proof. Assume the contrary and suppose that there exists a point z_0 in C then

$$|f(z)| \le |f(z_0)| \tag{1.17}$$

By Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz$$

If C is a circle with centre at z_0 and radius r,
then $C : |z - z_0| = r \Longleftrightarrow z - z_0 = re^{i\theta}, 0 \leq \theta \leq 2\pi$ so that $dz = ire^{i\theta}d\theta$ and $|dz| = rd\theta$
Thus

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}}ire^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})d\theta \\ |f(z_0)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})d\theta\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|d\theta \\ &\leq \frac{|f(z_0+re^{i\theta})|}{2\pi}\int_0^{2\pi} d\theta \end{aligned}$$

$$|f(z_0)| \leq |f(z_0+re^{i\theta})| \tag{1.18}$$

Since we suppose that $|f(z)| \leq |f(z_0)|$ that is,

$$|f(z_0+re^{i\theta})| \leq |f(z_0)| \tag{1.19}$$

Combining (1.18) and (1.19) gives $|f(z_0 + re^{i\theta})| = |f(z_0)|$ that is, $|f(z)| = |f(z_0)|$.
This is valid only when $f(z)$ is a constant function, which contradict, the hypothesis of the theorem.
Proof**(Minimum Modulus Theorem)**
If $f(z)$ is analytic within and on C and $f(z) \neq 0$ inside C. That is $\dfrac{1}{f(z)}$ is analytic within and on C. By maximum Modulus theorem $\dfrac{1}{|f(z)|}$ attains its maximum on C. Hence, $|f(z)|$ attains its minimum on C. □

Definition 1.5.1 If $z = z_0$ is a zero of order k of an analytic function $f(z)$ inside and on a simple closed curve C, then f can be expressed as

$$f(z) = (z - z_0)^k Q(z),$$

where $Q(z)$ is analytic at z_0 and $Q(z_0) \neq 0\ \forall k \in \mathbb{N}$.

Definition 1.5.2 If $z = z_0$ is a pole of order n of an analytic function $f(z)$ inside and on a simple closed curve C, then

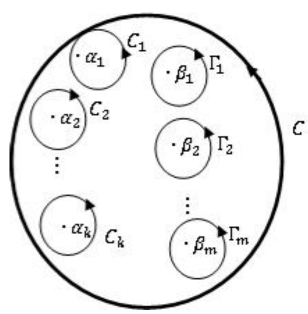
$$f(z) = \frac{Q(z)}{(z - z_0)^n},$$

where $Q(z)$ is an analytic at $z = z_0$ and $Q(z_0) \neq 0$.

Theorem 1.5.7 (The Argument Principle)
Let $f(z)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside and suppose that $f(z) \neq 0$ on C. If N and P are respectively, the number of the zero's and poles inside C, then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P$$

Proof. Let the zeros' of $f(z)$ be $\alpha_1, \alpha_2, \cdots, \alpha_k$ with respective orders $n_1, n_2, \cdots, n_k$ and that of poles of $f(z)$ are $\beta_1, \beta_2, \cdots, \beta_m$ with orders $p_1, p_2, \cdots, p_n$. Enclose each zero and pole by non overlapping circles $C_1, C_2, \cdots, C_k$ and $\Gamma_1, \Gamma_2, \cdots, \Gamma_m$ respectively.



$$f(z) = \frac{(z - \alpha_1)^{n_1}(z - \alpha_2)^{n_2} \cdots (z - \alpha_k)^{n_k}}{(z - \beta_1)^{p_1}(z - \beta_2)^{p_2} \cdots (z - \beta_m)^{p_m}} Q(z),$$

where $Q(z)$ is analytic inside and on C such that $f(z) \neq 0$.
Forming logarithmic derivative, we have

$$\begin{aligned} \ln f(z) &= n_1 \ln(z - \alpha_1) + n_2 \ln(z - \alpha_2) + \cdots + n_k \ln(z - \alpha_k) + \ln Q(z) \\ &-p_1 \ln(z - \beta_1) - p_2 \ln(z - \beta_2) - \cdots - p_m \ln(z - \beta_m) \end{aligned}$$

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{n_1}{z-\alpha_1}+\frac{n_2}{z-\alpha_2}+\cdots+\frac{n_k}{z-\alpha_k}+\frac{Q'(z)}{Q(z)}\\ &-\frac{p_1}{z-\beta_1}-\frac{p_2}{z-\beta_2}-\cdots-\frac{p_m}{z-\beta_m}\\ &=\textstyle\sum_{s=1}^{k}\displaystyle\frac{n_s}{z-\alpha_s}+\frac{Q'(z)}{Q(z)}-\textstyle\sum_{t=1}^{m}\displaystyle\frac{p_t}{z-\beta_t}\end{aligned}$$

Taking integral of both sides

$$\int_C \frac{f'(z)}{f(z)}dz = \int_C \left\{\sum_{s=1}^{k}\frac{n_s}{z-\alpha_s}+\frac{Q'(z)}{Q(z)}-\sum_{t=1}^{m}\frac{p_t}{z-\beta_t}\right\}dz$$

Term by term integration yields

$$\int_C \frac{f'(z)}{f(z)}dz = \int_C \sum_{s=1}^{k}\frac{n_s}{z-\alpha_s}dz+\int_C \frac{Q'(z)}{Q(z)}dz-\int_C \sum_{t=1}^{m}\frac{p_t}{z-\beta_t}dz$$

$$\int_C \frac{f'(z)}{f(z)}dz = \sum_{s=1}^{k}\int_C \frac{n_s}{z-\alpha_s}dz-\sum_{t=1}^{m}\int_C \frac{p_t}{z-\beta_t}dz+\int_C \frac{Q'(z)}{Q(z)}dz$$

That is,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\sum_{s=1}^{k}\int_C \frac{n_s}{z-\alpha_s}dz-\frac{1}{2\pi i}\sum_{t=1}^{m}\int_C \frac{p_t}{z-\beta_t}dz+\frac{1}{2\pi i}\int_C \frac{Q'(z)}{Q(z)}dz$$

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\sum_{s=1}^{k}n_s(2\pi i)-\frac{1}{2\pi i}\sum_{t=1}^{m}p_t(2\pi i)+\frac{1}{2\pi i}\int_C \frac{Q'(z)}{Q(z)}dz$$

That is,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \sum_{s=1}^{k}n_s-\sum_{t=1}^{m}p_t+\frac{1}{2\pi i}\int_C \frac{Q'(z)}{Q(z)}dz$$

Hence,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N-P+0$$

Since $\int_C \frac{Q'(z)}{Q(z)}dz = 0$ by Cauchy integral theorem.
Where N =total number of zeros
P =total number of poles inside C.

$\square$

Example 1.5.3 Evaluate

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$$

where $C : |z-1| = 2$ and $f(z) = \dfrac{z^2-16}{z(z-1)^2(z-4)}$

Solution:

Exercise 6

1. Find
$$I = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz, C : |z-1-i| = 2$$
for which
   (i) $f(z) = \dfrac{z-2}{z(z-1)}$
   (ii) $f(z) = \dfrac{z^2-9}{z^2+1}$
2. Evaluate
$$\int_C \frac{f'(z)}{f(z)}dz \text{ if } C : |z| = \pi$$
and
   (a) $f(z) = \sin \pi z$
   (b) $f(z) = \cos \pi z$
   (c) $f(z) = \tan \pi z$.
3. Evaluate
$$\int_{|z-\frac{\pi}{2}|} \tan z dz$$
4. Evaluate
$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$$
where $C : |z-1| = 5$ and $f(z) = \dfrac{z^2-16}{z(z-1)^2(z-4)}$

5. Evaluate
$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$$
where $C : |z-1| = 2$ and $f(z) = \dfrac{z^2+16}{z(z+1)^2(z-4)}$

6. Evaluate
$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$$
where $C : |z+1| = 3$ and $f(z) = \dfrac{z^2-16}{z(z-1)^2(z+4)}$

Chapter 2

Conformal and Bilinear or Fractional transformation (Mobius)

2.1 Complex Mapping

Definition 2.1.1 $f : Z \to W$ is the mapping that transforms points (x,y) in the z-plane onto points (u, v) in the w-plane.

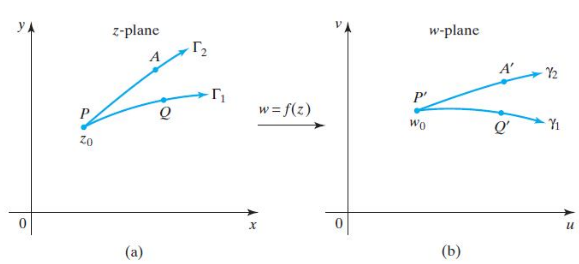


Figure 2.1: Mapping of curves Γ_1 and Γ_2 to γ_1 and γ_2 by $w = f(z)$

For example, $f(z) = 2z + i - 1 \Rightarrow$ the image of $z = 1 + 2i$ is

$$\begin{aligned} f(z) = w &= 2(1 + 2i) + i - 1 \\ &= 2 + 4i + i - 1 \\ &= 5i + 1 = (1, 5) \end{aligned}$$

Definition 2.1.2 $w = f(z)$ is said to be analytic at $z = z_0$, if it is continuous, differentiable at the point $z = z_0$.

Definition 2.1.3 A function $w = f(z)$ is said to be analytic at region R, if it is analytic at every point in $R\,\forall z_0 \in R$.

Example 2.1.1 If $f(z) = z^2 + 2z, z = 1$

$f'(z) = 2z + 2$ at $z = 1 \Rightarrow f'(1) = 4$. Hence, analytic at $z = 1$.

Example 2.1.2 $f(z) = \dfrac{1}{z-1}$ at $z = 1$. This is clearly not analytic at $z = 1$. if a function is not analytic is said to be Singular.

Definition 2.1.4 (Fixed point of a Function)
A point $z = z_0$ of a function $w = f(z)$ is said to be a fixed point if

$$f(z) = z$$

at that point.
For example, $f(z) = z^2 + 1$
$f(z) = z \Rightarrow z^2 + 1 = z$ that is

$z^2 - z + 1 = 0$

$z = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm \sqrt{3}i}{2}$ that is $z_1 = \dfrac{1}{2} + \dfrac{\sqrt{3}i}{2}$ and $z_2 = \dfrac{1}{2} - \dfrac{\sqrt{3}i}{2}$.

Definition 2.1.5 Geometric definition of conformal mappings We start with a somewhat hand-wavy definition:

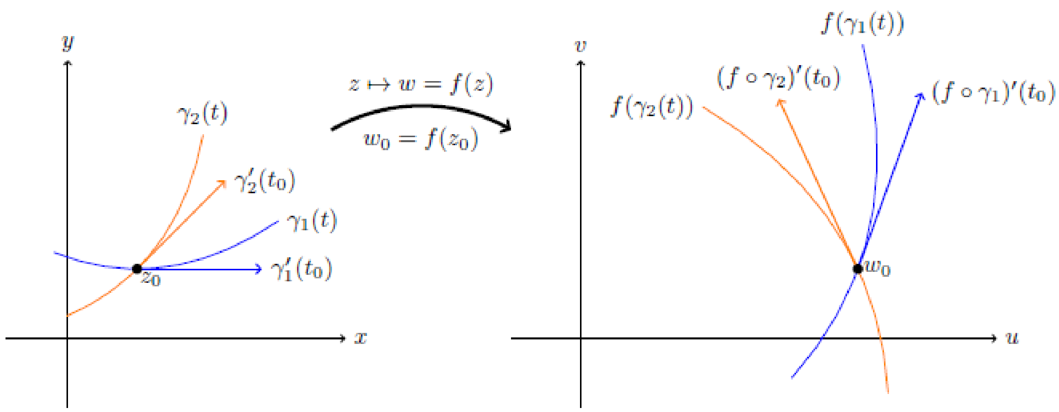
Informal definition. Conformal maps are functions on $\mathbb{C}$ that preserve the angles between curves.
More precisely: Suppose f(z) is differentiable at z_0 and $\gamma(t)$ is a smooth curve

through z_0. To be concrete, let's suppose $\gamma(t_0) = z_0$. The function maps the point z_0 to $w_0 = f(z_0)$ and the curve γ to $\bar{\gamma}(t) = f(\gamma(t))$. Under this map, the tangent vector $\gamma'(t_0)$ at z_0 is mapped to the tangent vector $\bar{\gamma}'(t_0) = (f \circ \gamma)'(t_0)$ at w_0. With these notations we have the following definitions.

Definition 2.1.6 The function $f(z)$ is conformal at z_0 if, there is an angle ϕ and a scale $a > 0$ such that for any smooth curve $\gamma(t)$ through z_0. The map f rotates the tangent vector at z_0 by ϕ and scales it by a. That is, for any γ, the tangent vector $(f \circ \gamma)'(t_0)$ is found by rotating $\bar{\gamma}'(t_0)$ by ϕ and scaling it by a.

If $f(z)$ is defined on a region A, we say it is a conformal map on A if it is conformal at each point z in A.
Note. The scale factor a and rotation angle ϕ depends on the point z, but not on any of the curves through z.

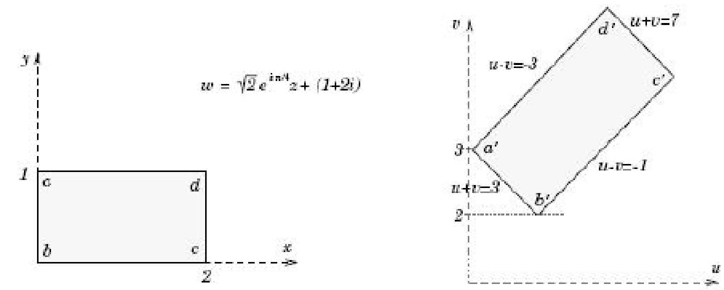
Example 2.1.3 The figure below shows a conformal map $f(z)$ mapping two curves through z_0 to two curves through $w_0 = f(z_0)$. The tangent vectors to each of the original curves are both rotated and scaled by the same amount.



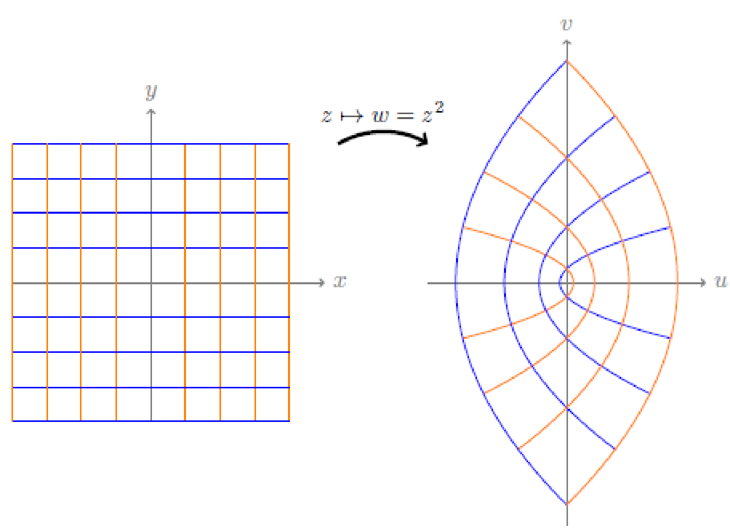
A conformal map rotates and scales all tangent vectors at z_0 by the same amount.
Remark
1. Conformality is a local phenomenon. At a different point z_1 the rotation angle and scale factor might be different.
2. Since rotations preserve the angles between vectors, a key property of conformal maps is that they preserve the angles between curves

Example 2.1.4 Let D be the rectangular region in the z plane bounded by $x = 0, y = 0, x = 2$ and $y = 1$. Find the image of D under the transformation

$w = (1+i)z + (1+2i)$.
SOLUTION:



Example 2.1.5 Show that $f(z) = z^2$ maps horizontal and vertical grid lines to mutually orthogonal parabolas. We will see that $f(z)$ is conformal.
SOLUTION:



Theorem 2.1.1 $w = f(z)$ is transformation where $f(z)$ is analytic at $z = z_0$ and $f'(z_0) \neq 0$, then, under the transformation the tangent at z_0 to any curve C in z-plane passing through z_0 is rotated through the angle $arg f'(z_0)$.

Proof. Let $w = t(z)$ be a transformation such that $f'(z) \neq 0$ That is, it is different from zero.

$$\frac{dw}{dt} = \frac{dw}{dz} \cdot \frac{dz}{dt},$$

where t is a parameter. That is,

$$z = z(t)\{x = x(t), y = y(t)\}$$

and the corresponding value of $w = w(t)\{u = u(t), v = v(t)\}$ in this case $\frac{dw}{dt}, \frac{dz}{dt}$ are tangents to the curve C and $C*$ respectively.

$$\Rightarrow \frac{dw}{dt} = f'(z) \cdot \frac{dz}{dt}$$

at $z = z_0$

$$\Rightarrow \frac{dw}{dt}\,|_{z=z_0} = f'(z_0) \cdot \frac{dz}{dt}\,|_{z=z_0}$$

Let

$$\frac{dw}{dt} = \rho_0 e^{i\phi_0}, f'(z_0) = Re^{i\alpha}, \frac{dz}{dt} = Pe^{i\theta_0}$$

$$\Rightarrow \rho_0 e^{i\phi_0} = Re^{i\alpha} \cdot Pe^{i\theta_0} = RPe^{i(\theta_0 + \alpha)}$$

$$\Rightarrow \phi_0 = \theta_0 + \alpha$$

Hence, the curve C has been rotated by $arg f'(z_0) = \alpha$. Since every curve in C is rotated through the same angle in the w-plane, thus angle are preserved both in magnitude and in the sense of rotation. □

Recall: If $w = f(z)$, then

$$\frac{\partial(u, v)}{\partial(x, y)} = |f'(z)|^2$$

$|f'(z)|^2$ is called the magnification factor, $\frac{\partial(u, v)}{\partial(x, y)}$ is called the Jacobian of the transformation.

If $w = f(z)$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{vmatrix}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$= \left|\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right| = |f'(z)|^2$$

For example, If $w = z^2 - 2z$, find the Jacobian.
Solution:

$$u + iv = (x + iy)^2 - 2(x + iy)$$

which gives

$$u = x^2 - y^2 - 2x, \; v = 2xy - 2y$$

$$\frac{\partial u}{\partial x} = 2x - 2, \frac{\partial u}{\partial y} = -2y, \frac{\partial v}{\partial x} = 2y, \frac{\partial v}{\partial y} = 2x - 2$$

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x-2 & 2y \\ -2y & 2x-2 \end{vmatrix} = (2x-2)^2 + (2y)^2 = 4[(x-1)^2 + y^2]$$

$$f'(z) = 2z - 2 = 2(x + iy) - 2 = (2x - 2) + 2iy$$

$$|f'(z)|^2 = (2x - 2)^2 + (2y)^2 = 4[(x - 1)^2 + y^2]$$

the same as obtained above.

2.1.1 Types of Transformations

1. If $w = z + \alpha$ where α is a complex constant the transformation is called translation. that is, the figure is displaced (or translated) by α on the w-plane.

2. If $w = \alpha z$, this transformation is called stretching if $|\alpha| > 1$, and is called subtraction if $|\alpha| < 1$.

3. If $w = e\imath\theta_0 z$, this transformation is called rotation on the w-plane.

4. If $w = \dfrac{1}{z}$, is called inversion.

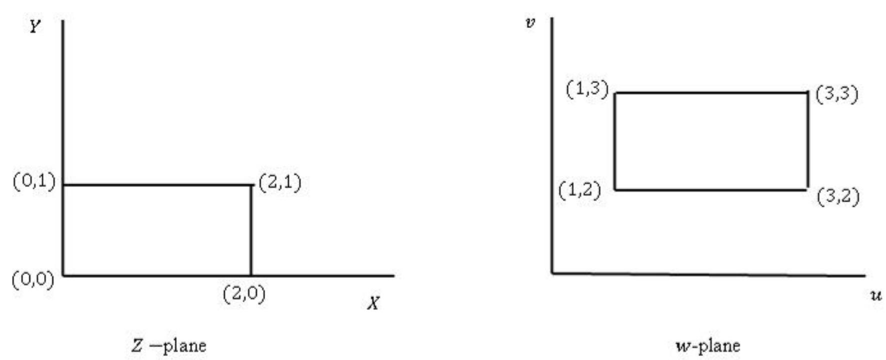
Example 2.1.6 Find the image of the rectangular region R in the z-plane bounded by $x = 0, y = 0, x = 2, y = 1$ under the transformations

(i) $w = z + (1 + 2i)$

(ii) $w = \sqrt{2}e^{i\frac{\pi}{4}}z$

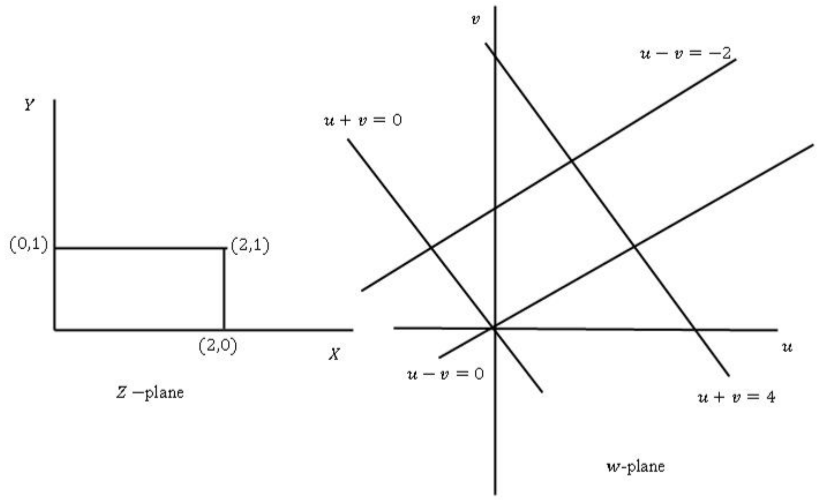
(iii) $\dfrac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}z + (1 - 2i)$

Solution:



(i)

(ii) See the figure below



(iii) Exercise.

Theorem 2.1.2 If $f(z)$ is conformal at z_0 then there is a complex number $c = ae^{i\phi}$ such that the map f multiplies tangent vectors at z_0 by c. Conversely, if the map f multiplies all tangent vectors at z_0 by $c = ae^{i\phi}$ then f is conformal at z_0.

Proof. By definition f is conformal at z_o means that there is an angle ϕ and a scalar $a > 0$ such that the map f rotates tangent vectors at z_0 by ϕ and scales them by a. This is exactly the effect of multiplication by $c = ae^{i\phi}$.

□

2.2 Bilinear or Fractional transformation (Mobius)

Definition 2.2.1 A fractional linear transformation is a mapping of the form

$$w = \frac{az + b}{cz + d}$$

where a, b, c, d are complex constants and $ad - bc \neq 0$.
These are also called Mobius transforms or bilinear transforms.
The above transformation has the following properties

(i) mapping circles onto circles.

(ii) planes(lines) are mapped onto planes(lines)

(iii) planes (lines) are mapped onto disk(circles)

(iv) circles (disk) are mapped onto planes (lines)

Example 2.2.1 Find a bilinear transformation which maps the point $z = 0, -i, -1$ (in z-plane) onto the points $w = i, 1, 0$ in the w-plane respectively.
Solution:

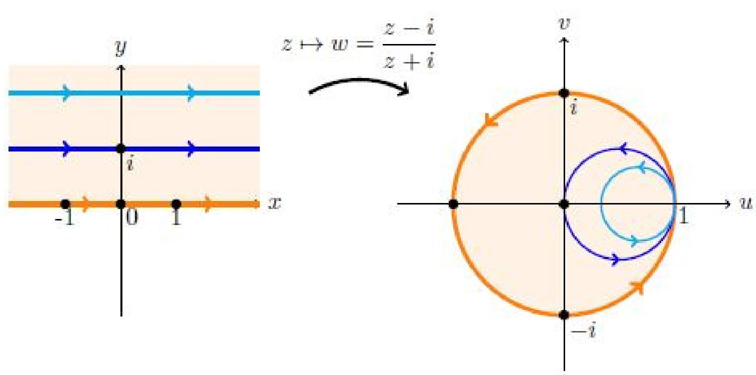
Example 2.2.2 Find the image of the circle $|z| = 2$ under the transformation

(i) $w = \dfrac{z-i}{z+i}$

(ii) $w = z + 2i$

Solution:
(i)
(ii) Exercise.

Example 2.2.3 Show that transformation $w = \dfrac{z-i}{z+i}$, maps the x-axis to the unit circle and the upper half-plane to the unit disk.
Solution:



Theorem 2.2.1 A linear fractional transformation maps lines and circles to lines and circles. Before proving this, note that it does not say lines are mapped to lines and circles to circles. For example, in Example above the real axis is mapped the unit circle.

Proof. We start by showing that inversion maps lines and circles to lines and circles. Given z and $w = \frac{1}{z}$ we define x, y, u and v by

$$z = x + iy \quad \text{and} \quad w = \frac{1}{z} = \frac{x - iy}{x^2 + y^2} = u + iv$$

So, $u = \frac{x}{x^2 + y^2}$ and $v = \frac{-y}{x^2 + y^2}$. Now, every circle or line can be described by the equation

$$Ax + By + C(x^2 + y^2) = D$$

(If $C = 0$ it describes a line, otherwise a circle.) We convert this to an equation in u, v as follows.

$$Ax+By+C(x^2+y^2) = D \Leftrightarrow \frac{Ax}{x^2 + y^2}+\frac{By}{x^2 + y^2}+C = \frac{D}{x^2 + y^2} \Leftrightarrow Au+Bv+C = D(u^2+v^2)$$

In the last step we used the fact the $u^2 + v^2 = |w|^2 = \frac{1}{|z|^2} = \frac{1}{x^2 + y^2}$. We have shown that a line or circle in x, y is transformed to a line or circle in u, v. This shows that inversion maps lines and circles to lines and circles.
We note that for the inversion $w = \frac{1}{z}$.

1. Any line not through the origin is mapped to a circle through the origin.
2. Any line through the origin is mapped to a line through the origin.
3. Any circle not through the origin is mapped to a circle not through the origin.
4. Any circle through the origin is mapped to a line not through the origin.

Now, to prove that an arbitrary fractional linear transformation maps lines and circles to lines and circles, we factor it into a sequence of simpler transformations.

First suppose that $c = 0$. So, $w = (az + b)/d$. Since this is just translation, scaling and rotating, it is clear it maps circles to circles and lines to lines.

Now suppose that $c \neq 0$. Then,

$$w = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) + b - \frac{ad}{c}}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}$$

So, $w = f(z)$ can be computed as a composition of transforms

$$z \to w_1 = cz + d \to w_2 = \frac{1}{w_1} \to w = \frac{a}{c} + (b - ad/c)w_2$$

We know that each of the transforms in this sequence maps lines and circles to lines and circles. Therefore the entire sequence does also. QED

□

Exercise 7

1. Given a triangle T in z-plane with vertices, $i, 1-i, 1+i$ respectively. Determine the transformations
   (a) $w = 3z + 4 - 2i$
   (b) $w = 2e^{i\frac{\pi}{2}}z$
2. Find the image of the circle $|z| = 1$ under the transformation $w = \dfrac{1}{z-1}$.
3. (i) Define fixed point of a complex function, hence obtain the fixed points of $f(z) = z^2 - 3z$
   (ii)Find the image of the rectangular region R bounded by $x = 0, y = 0, x = 5$ and $y = 3$, under the transformation $w = \sqrt{2}e^{i\frac{\pi}{4}}z$.
4. Find the image of the circle $|z| = 2$ under the transformation $w = z + \dfrac{8}{z}$.
5. (i) Define fixed point of a complex function, hence obtain the fixed points of $f(z) = z^2 + 3z$
   (ii)Find the image of the rectangular region R bounded by $x = 0, y = 0, x = 3$ and $y = 5$, under the transformation $w = \sqrt{2}e^{i\frac{\pi}{4}}z$.
6. Find the image of the circle $|z| = 2$ under the transformation $w = z + \dfrac{16}{z}$.

7. (i) Define fixed point of a complex function, hence obtain the fixed points of $f(z) = z^2 - 3z$
(ii)Find the image of the rectangular region R bounded by $x = 0, y = 0, x = -5$ and $y = -3$, under the transformation $w = \sqrt{2}e^{i\frac{\pi}{4}}z$.

8. Find the image of the circle $|z| = 1$ under the transformation $w = z + \dfrac{32}{z}$.

9. (i) Define a conformal transformations.
(ii) If R is the rectangular region bounded by $x = 0, y = 0, x = 2$ and $y = 5$. Describe the image of R under the transformation $w = (1 + i)z + (3 + 4i)$.

10. Find the image of the circle $|z| = 2$ under the transformation $w = \dfrac{z + 1}{z - 1}$.

11. (i) Define a conformal transformations.
(ii) If R is the rectangular region bounded by $x = 0, y = 0, x = -1$ and $y = 3$. Describe the image of R under the transformation $w = (1 - i)z + (3 - 4i)$.

12. Find the image of the circle $|z| = 2$ under the transformation $w = \dfrac{z - i}{z + i}$.

13. (i) Define a conformal transformations.
(ii) If R is the rectangular region bounded by $x = 1, y = 0, x = -2$ and $y = 5$. Describe the image of R under the transformation $w = (1 + i)z + (3 - 4i)$.

14. Find the image of the circle $|z| = 2$ under the transformation $w = \dfrac{z - 1}{z + 1}$.

15. Describe the effect of the linear transformation $w = 2iz+3$ when mapping geometrical shapes from the z-plane onto the w-plane. Sketch the image of the rectangle in the z-plane with its corners at $(1, 1), (3, 1), (3, 2)$, and $(1, 2)$, and show the correspondence between corners in the two planes.

16. Describe the effect of the linear transformation $w = (1 + i)z - i$ when mapping geometrical shapes from the z-plane to the w-plane. Sketch (a) the image of the unit circle $|z| = 1$ and
(b) the image of the ellipse $(x - 3)2/9 + y2/4 = 1$. In each case show how four points on the curve in the z-plane map to the w-plane.

17. Find a linear transformation that maps the triangle with its vertices A, B, and C at points $0, 1+i$, and $2-i$ in the z-plane onto the similar triangle with vertices $A*, B*$, and $C*$ at $1-i, 5-i$, and $3-7i$ in the w-plane.

18. Find the linear transformation with the fixed point $2-i$ that maps $z=-i$ to $w=2-3i$.

19. Find the linear transformation with the fixed point $3+2i$ that maps $z=1$ to $w=-7$.

20. In the following transformations find the fixed point $z*$ when one exists, the angle of rotation α about $z*$ that is introduced, and the magnification factor ρ:
(a)$w=2z+1-3i$. (b) $w=iz+4$. (c) $w=z+1-2i$.

21. Find a linear transformation $w=az+b$ that maps the infinite strip $k<y<k+h$ in the z-plane onto the strip $0<u<1$ in the w-plane in such a way that $w(ik)=0$.

22. . Find a linear transformation $w=az+b$ that maps the infinite strip $k<x<k+h$ in the z-plane onto the strip $0<u<1$ in the w-plane in such a way that $w(k)=0$.

23. Given that $w=1/z$, find the image in the w-plane of the family of parallel straight lines $y=x+c$ in the z-plane.

24. By using the symmetry properties of linear fractional mappings, or otherwise, find how $w=z/(z-1)$ maps the annulus $1 \leq |z| \leq 2$ in the z-plane onto the w-plane.

Theorem 2.2.2 Three points z_1, z_2 and z_3 can always be mapped onto three prescribed points; w_1, w_2 and w_3 respectively by one (and only one) linear transformation $w=f(z)$, this mapping is given by

$$\frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1} = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}$$

Proof: Exercise.

NOTE: If one of these points is the point ∞, the quotient of the two difference containing this point must be replaced by 1. That is, the above ratio becomes

$$\frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1} = \frac{z_2-z_3}{z-z_3} \text{ for } z_1=\infty$$

$$\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1}=\frac{z-z_1}{z-z_3} \text{ for } z_2=\infty$$
$$\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1}=\frac{z_2-z_1}{z_2-z_1} \text{ for } z_3=\infty$$

Similarly, for w_1, w_2, w_3 equals ∞ we have

$$\frac{w_2-w_3}{w-w_3}=\frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1} \text{ for } w_1=\infty$$

$$\frac{w-w_1}{w-w_3}=\frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1} \text{ for } w_2=\infty$$
$$\frac{w-w_1}{w_2-w_1}=\frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1} \text{ for } w_3=\infty$$

Example 2.2.4 Find a bilinear transformation which maps points $z=0,-i,-1$ onto $w=i,1,0$.
Solution:

Example 2.2.5 Find a linear transformation which maps $\infty, 0, 1$ in z-plane onto the points $1, i, -1$ respectively in the w-plane.
Solution:

Example 2.2.6 Find the fractional transformation which maps the points $z_1=-2, z_2=0, z_3=2$ onto the points $w_1=\infty, w_2=\frac{1}{2}, w_3=\frac{3}{4}$.
Solution:

Example 2.2.7 Find the image of the circle $|z|=1$ under the transformation $w=z+\frac{4}{z}$.
Solution:

Exercise 8

1. Find the image of $|z-1|=1$ under the transformation $w=\frac{1}{z}$.

2. Show that $w = \dfrac{1}{z}$ maps the circle $|z-3| = 5$ onto the circle $\left||w + \dfrac{3}{16}\right| = \dfrac{5}{16}$.

3. Show that $w = \dfrac{(z-i)}{(iz-1)}$ maps $Im(z) \geq 0$ onto $|w| \leq 1$

4. State the the transformation which maps the points z_1, z_2 and z_3 in z -plane on to the points w_1, w_2 and w_3 in w -plane respectively. Hence obtain a bilinear transformation for $z = \infty, 0, 1$ in z -plane onto $w = 1, i, -1$ in w-plane.

5. State the the transformation which maps the points z_1, z_2 and z_3 in z -plane on to the points w_1, w_2 and w_3 in w -plane respectively. Hence obtain a bilinear transformation for $z = 1, 0, \infty$ in z -plane onto $w = -1, i, 1$ in w-plane.

6. State the the transformation which maps the points z_1, z_2 and z_3 in z -plane on to the points w_1, w_2 and w_3 in w -plane respectively. Hence obtain a bilinear transformation for $z = 1, i, -1$ in z -plane onto $w = \infty, 0, 1$ in w-plane.

7. Map points $z_1 = i, z_2 = -i$, and $z_3 = 1$ onto the points $w_1 = -1, w_2 = 1$, and $w_3 = \infty$.

8. Map the points $z_1 = -1, z_2 = -i$, and $z_3 = 1$ onto the points $w_1 = -3 + i, w_2 = (2-4i)/5$, and $w_3 = 1 + i/3$.

9. Map the points $z_1 = 1, z_2 = 2 + i$, and $z_3 = i$ onto the points $w_1 = i, w_2 = (-1+2i)/5$, and $w_3 = 1/3$.

10. Map the points $z_1 = -1, z_2 = 1$, and $z_3 = \infty$ onto the points $w_1 = i, w_2 = -i$, and $w_3 = 1$.

11. Prove that the function $w = exp(\frac{\pi z}{a})$ maps the infinite strip of width a in the z-plane shown in the diagram on the left of Fig. 2.1 onto the upper half of the w-plane in the manner shown in the diagram on the right. Determine the images in the w-plane of the lines $x = c$ and $y = k$.

12. Prove that the function $w = sin(\pi z/a)$ maps the semi-infinite strip of width a in the z-plane shown in the diagram on the left of Fig. 2.2 onto the upper half of the w-plane in the manner shown in the diagram on

the right. Determine the images in the w-plane of the lines $x = c$ and $y = k$.

13. Prove that the function $w = cos(\pi z/a)$ maps the semi-infinite strip of width a in the z-plane shown in the diagram on the left of Fig. 2.3 onto the upper-half of the w-plane in the manner shown in the diagram on the right. Determine the images in the w-plane of the lines $x = c$ and $y = k$.

14. Prove that the function $w = cosh(\pi z/a)$ maps the semi-infinite strip of width a in the z-plane shown in the diagram on the left of Fig. 2.4 onto the upper half of the w-plane in the manner shown in the diagram on the right. Determine the images in the w-plane of the lines $x = c$ and $y = k$.

15. Prove that the function $w = \left(\frac{1+z}{1-z}\right)^2$ maps the interior of the unit semicircle in the z-plane in the diagram on the left of Fig. 2.5 onto the upper half of the w-plane in the manner shown in the diagram on the right.

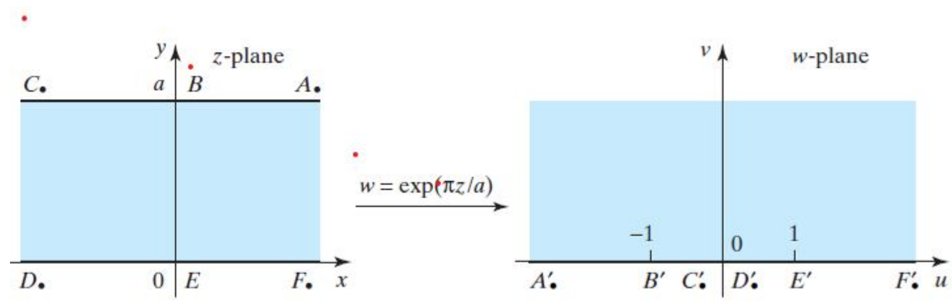


Figure 2.2: graph of $w = exp(\frac{\pi z}{a})$

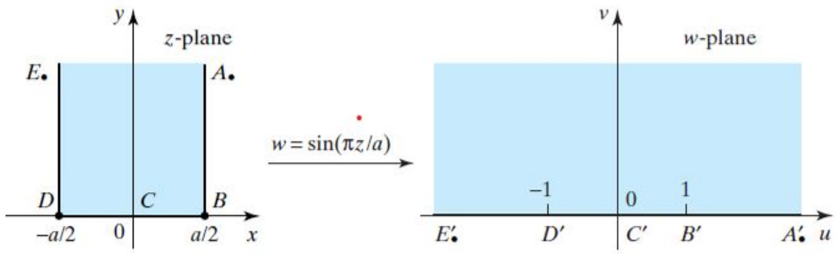


Figure 2.3: graph of $w = sin(\pi z/a)$

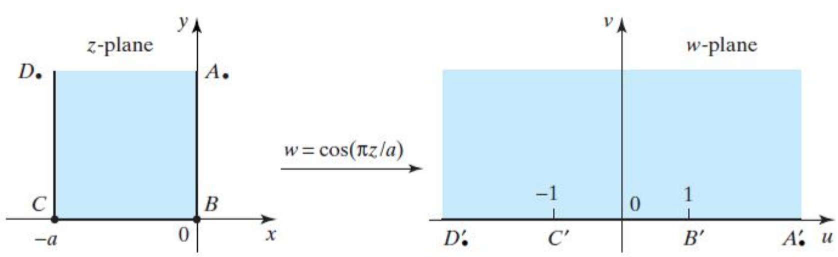


Figure 2.4: graph of $w = cos(\pi z/a)$

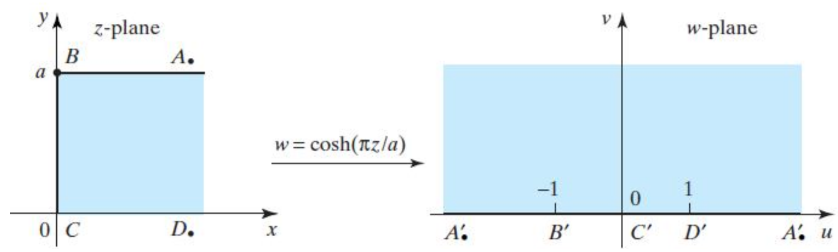


Figure 2.5: graph of $w = cosh(\pi z/a)$

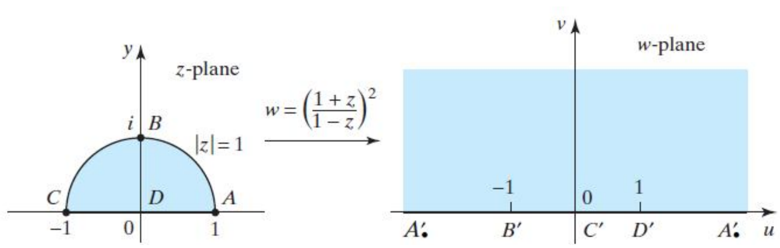


Figure 2.6: graph of $w = \left(\dfrac{1+z}{1-z}\right)^2$

16. Given that $w = z + k/z$, with k real, find the image in the w-plane of the lines $x = c$ and $y = d$. Find the values of k and R such that for given real a and b the transformation will map the circle $|z| = R$ onto the ellipse $\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$ in the w-plane.

17. Verify that $w = \dfrac{k(z - z_0)}{(z - \bar{z}_0)}$, with $|k| = 1$ and z_0 an arbitrary point in the upper half of the z-plane, maps the upper half of the z-plane onto $|w| < 1$ and z_0 to the point $w = 0$.

18. Verify that $w = k\left(\dfrac{z - z_0}{\bar{z}_0 z - 1}\right)$, with $|k| = 1$ and z_0 an arbitrary point such that $|z_0| < 1$, maps $|z| < 1$ onto $|w| < 1$ and z_0 to the point $w = 0$.