EXAMINATION

ABIMBOLA

₦250

MATH104

NOTES/HINTS

ROMAN3:23 For all have Sinned and fall short of the glory of God

ROMAN6:23 For the wages of sin is death but the gift of God is eternal life in Christ Jesus our Lord

JOHN3:16 For God so Loved the world that He gave His only begotten Son that whoever believes in Him should not perish but have everlasting life

ROMAN10:9 That if you Confess with your mouth the Lord Jesus and believe in your heart that God raised Him from the dead you will be saved

MATHEW13:9 He who has ears to hear, Let him hear!

PASTQUESTIONS

EXERCISES

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BAPTIST STUDENT FELLOWSHIP

TRUTH: A wise son heareth his father's instruction but a scorner heareth not rebuke (Proverb 13:1)

COURSE OUTLINE

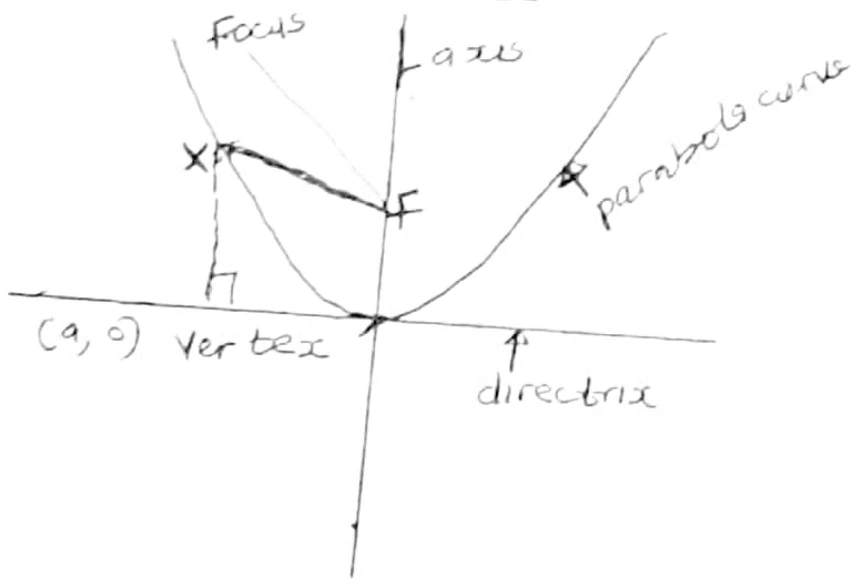
- ▷ CONIC SECTION
- ▷ RATE AND PROPORTION
- ▷ ERROR AND APPROXIMATION
- ▷ SOLUTION OF NON-LINEAR EQUATION
- ▷ MOTION UNDER GRAVITY
- ▷ MAXIMA AND MINIMA
- ▷ SOLUTION OF ORDINARY DIFFERENTIAL EQN
- ▷ AREA UNDER CURVE
- ▷ VOLUME OF REVOLUTION

CONIC SECTION

Conic section is made up of
I) Parabola (II) hyperbola (III) ellipse
(IV) Rectangular hyperbola

1

PARABOLA



DEFINITION OF TERMS

PARABOLA: is the set of points in a plane that are equidistant from a fixed point F (called the focus) and a fixed line [called the directrix]

VERTEX: is the point halfway between the focus and directrix that lies on the parabola.

AXIS: The line through the focus perpendicular to the directrix is called axis of the parabola

ECCENTRICITY: This is the ratio of the distance measured from the focus to a point X on the curve and perpendicular distance from directrix to that point X.

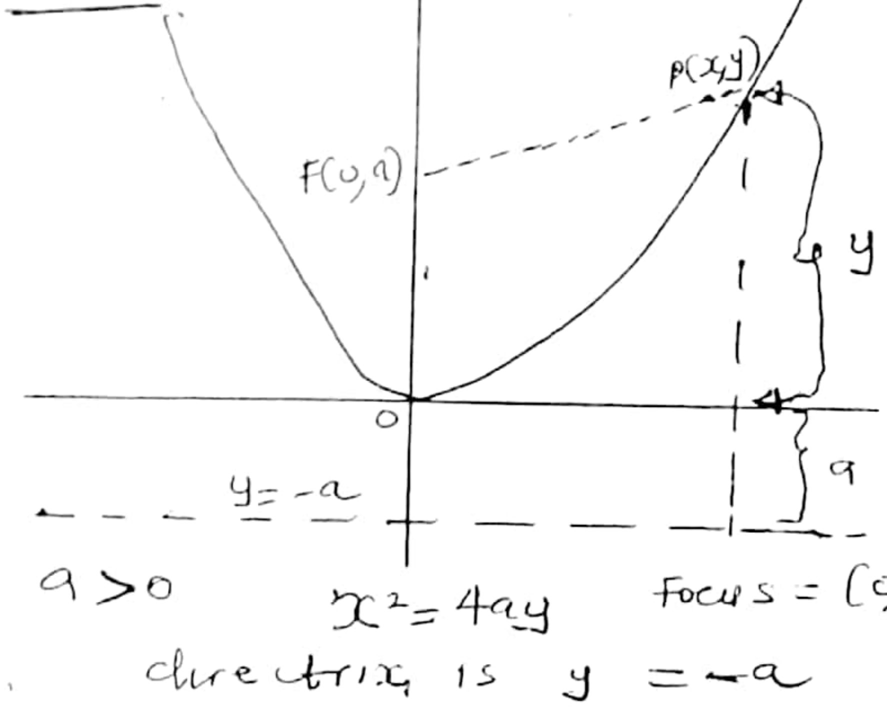
$$e = \frac{|\text{Focus}|}{|\text{directrix}|}$$

For parabola distance from the centre to a focus equal distance from centre to vertex. Eccentricity of a parabola equal one which means the distance from a fixed point equal to the distance from a fixed line.

Eccentricity	CONIC SECTION
$e = 1$	parabola
$e < 1$	ellipse
$e > 1$	hyperbola
$e = \pi$	rectangular hyperbola

EQUATION OF PARABOLA

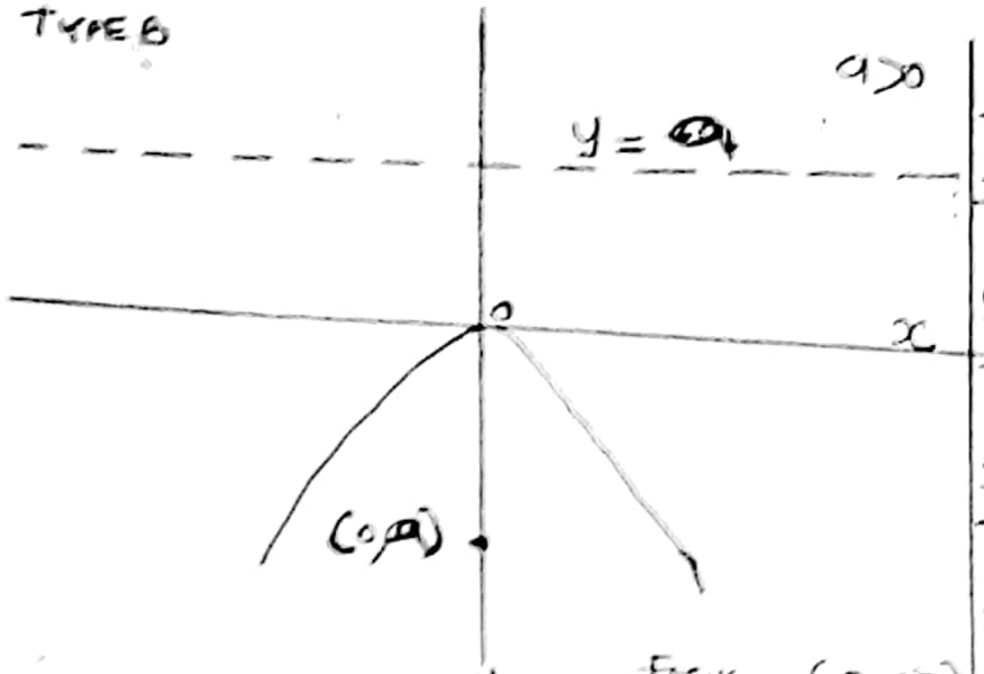
TYPE A



$a > 0$ $\quad x^2 = 4ay$ $\quad$ Focus = $(0, a)$

directrix is $y = -a$

TYPE B

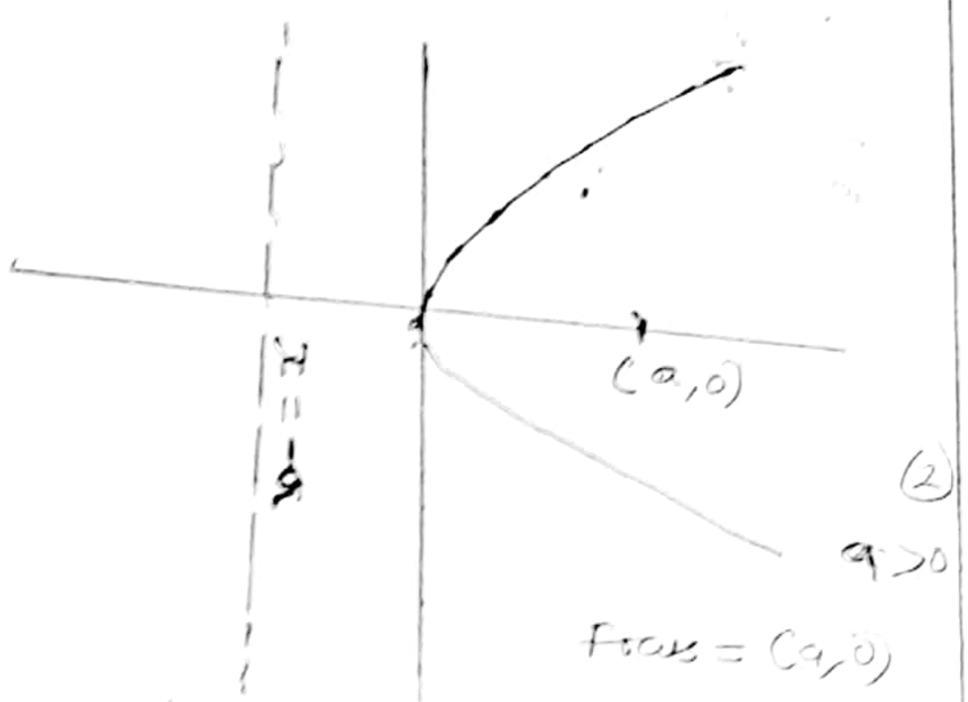


$x^2 = -4ay$

directrix is $y = a$ $\quad a > 0$

PARABOLA	DIRECTRIX	EQN OF LACTUM RECTUM	Coord of L[illegible]
$y^2 = 4ax$	$x = -a$	$x = a$	$(a,$ [illegible]
$y^2 = -4ax$	$x = a$	$x = -a$	$(a,$ [illegible]
$x^2 = 4ay$	$y = -a$	$y = a$	$(2a, a)$
$x^2 = -4ay$	$y = a$	$y = -a$	$(-2a, a)$
Vertex $(0, 0)$	Length of lactum rectum $= 4a$	Focus for $x^2 = \pm 4ay$ $(0, \pm a)$	Focus for $y^2 = \pm 4ax$ $(\pm a, 0)$

TYPE C

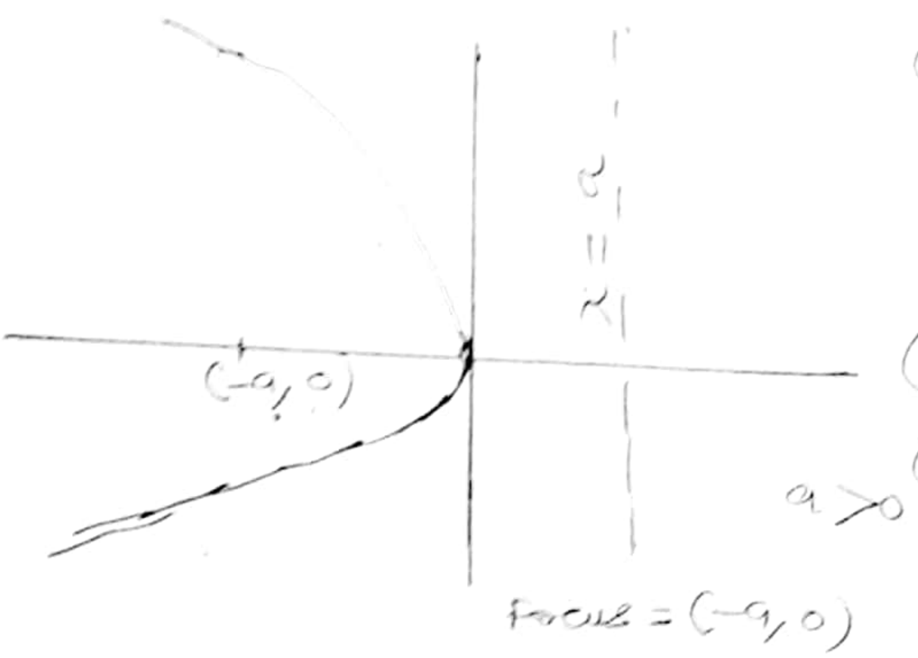


directrix is $x = -a$

Equation of parabola is $y^2 = 4ax$

(2)

TYPE D



directrix is $x = a$

Equation of parabola is

$y^2 = -4ax$

(1) Equation of tangent is

$$y - y_1 = m(x - x_1)$$

Point of contact of parabola and tangent is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

Equation of tangent is also gotten as $yy_1 = 2a(x + x_1)$ at (x_1, y_1)

(2) Equation of Normal is

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

Point of contact of normal and parabola is $(am^2, -2am)$

Equation of normal is also gotten as $\frac{y - y_1}{y_1} = \frac{-1}{2a}(x - x_1)$

(3) Given focus $= (x_1, y_1)$ and a point coordinate (x_2, y_2). The vertex $(\bar{x}, \bar{y})$ is given as $(\bar{x}, \bar{y}) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

(4) $|\text{Focus}| = |\text{directrix}|$

(5) Equation of tangent at point of contact $\Rightarrow y = mx - \frac{a}{m} + \frac{2a}{m}$

$$\boxed{\begin{aligned} y &= mx + \frac{a}{m} \qquad c = a/m \\ y &= mx + c \end{aligned}}$$

(6) Equation of normal at point of contact is given as

$$y = \frac{2a}{m} + \frac{am^3}{1} - \frac{x}{m}$$

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TRUTH: He that keepeth his mouth keepeth his life (Proverb 13:3)

EXAMPLE

Given a parabola $y^2 = 12x$. Find the Focus and the directrix given that the vertex is at the origin

SOLUTION

$y^2 = 12x$ --- Given

Compare with $y^2 = 4ax$ --- standard

$4a = 12 \quad a = 12/4 = 3$

Focus $= (a, 0) = (3, 0)$

directrix is $x = -a$, $x = -3$

EXAMPLE

The parabola $x^2 = -6y$ find the focus and directrix

SOLUTION

$x^2 = -6y$ -- Given

Compare with $x^2 = -4ay$ --- standard

$-4a = -6 \quad a = 6/4 = 3/2$

Focus $= (0, -a) = (0, -3/2)$

directrix $y = a$, $y = 3/2$

EXAMPLE

Find the equation of parabola with its vertex at $(3, 2)$ and Focus $(5, 2)$

(b) Find the directrix of the equation

SOLUTION

Let vertex $= (\bar{x}, \bar{y}) = (3, 2)$

Let coordinate point $P = (x_1, y_1)$

Let Focus $= (x_2, y_2) = (5, 2)$

Vertex is the mid point between the coordinate point and the focus i.e

$(\bar{x}, \bar{y}) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$(3, 2) = \left(\frac{x_1 + 5}{2}, \frac{y_1 + 2}{2}\right)$

$\frac{x_1 + 5}{2} = 3$ and $\frac{y_1 + 2}{2} = 2$

$x_1 + 5 = 6$ and $y_1 + 2 = 4$

$x_1 = 6 - 5 \quad y_1 = 4 - 2$

$x_1 = 1$ and $y_1 = 2$

Coordinate point $P(x_1, y_1) = (1, 2)$

The cartesian of focus $(x_2, y_2) = (5, 2)$

directrix is the point of difference between the coordinate point and focus. In above case the common point is 2

directrix is $x = 1$ or $x - 1 = 0$

distance of focus $= \sqrt{(x - x_2) + y - y_2}$

$|\text{Focus}| = \sqrt{(x-5)^2 + (y-2)^2}$

distance of directrix $= x - 1$

$|\text{directrix}| = x - 1$

For parabola

$|\text{Focus}| = |\text{directrix}|$

$\sqrt{(x-5)^2 + (y-2)^2} = x - 1$

Square both side

$(x-5)^2 + (y-2)^2 = (x-1)^2$

$\cancel{x^2} + 25 - 10x + y^2 + 4 - 4y = \cancel{x^2} + 1 - 2x$

$y^2 - 4y - 10x + 2x + 25 + 4 - 1 = 0$

$y^2 - 4y - 8x + 28 = 0$ OR

$y^2 - 4y = 8x - 28$

Find half of the coefficient of y, square it, add to L.H.S and subtract it again

$y^2 - 4y + 2^2 - 2^2 = 8x - 28$

$y^2 - 4y + 4 = 8x - 28 + 4$

$(y-2)^2 = 8x - 24$

$(y-2)^2 = 8(x-3)$ --- derived

$y^2 = 4ax$ --- standard

Length of latum rectum $= 4a = 8$

focal length $= a = 8/4 = 2$

Allter: Length of Lactum rectum is the absolute value of the variable with lower degree e.g

$$y^2 - 4x - 8x + 28 = 0$$

Variable with highest power or degree is y e.g y^2

Variable with lowest power is x e.g $-8x$

Lactum rectum length $= |-8| = 8$ //

vertex $= (0\ 0) = (\bar{X}, \bar{Y})$

$\bar{X} = x - x_0 = 0$ and $\bar{Y} = y - y_0 = 0$

From the equation derived

$x - 3 = 0$ and $y - 2 = 0$

$x = 3$ and $y = 2$

Vertex $= (3, 2)$

EXAMPLE

Find the equation of the tangent at the vertex of parabola represented as $x^2 + 4x - 8y + 4 = 0$

SOLUTION

$$x^2 + 4x + 4 = 0 + 8y$$

$$(x+2)^2 = 8y \text{ --- derived}$$

$$x^2 = 4ay$$

Length of lactum rectum $= 4a = 8$

(4)

Focal length $= a = 8/4 = 2$

Vertex $= (0, 0) = (\bar{X}, \bar{Y})$

$x + 2 = 0$, $y = 0$

$x = -2$ and $y = 0$ $(x, y) = (-2, 0)$

$$x^2 + 4x + 4 = 8y$$

differentiate w.r.t x

$$2x + 4 + 0 = 8\frac{dy}{dx}$$

$$m = \frac{dy}{dx} = \frac{2x+4}{8} = \frac{2(-2)+4}{8}$$

$$m = \frac{-4+4}{8} = 0$$

$y - y_1 = m(x + x_1)$ as equation of tangent

$$y - 0 = 0(x+2)$$

$$y - 0 = 0$$

$$y = 0$$

EXAMPLE

Find the equation of tangent and normal to $y^2 = 4ax$ at the end of lactum rectum

SOLUTION

The coordinate of the end of lactum rectum for $y^2 = 4ax$ is $(a, 2a)$

$$y^2 = 4ax$$

differentiate with respect to x

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y} = \frac{4a}{2(2a)} = 1$$

Equation of tangent $\Rightarrow y - y_1 = m(x - x_1)$

$$= y - 2a = 1(x - a)$$

$$y - 2a = x - a$$

$$y = x - a + 2a$$

$$y = x + a$$

Method II

Equation of tangent $= yy_1 = 2a(x + x_1)$

$$2a \times y = 2a(x + a)$$

$$y = x + a$$

Equation of normal $= y - y_1 = \frac{-1}{m}(x - x_1)$

$$= y - 2a = \frac{-1}{1}(x - a)$$

$$y - 2a = -x + a$$

$$y = -x + a + 2a$$

$$y = -x + 3a$$

Method II

Equation of normal is given as

$$\frac{y - y_1}{y_1} = \frac{-1}{2a}(x - x_1)$$

$$\frac{y - 2a}{2a} = \frac{-1}{2a}(x - a)$$

$$y - 2a = -x + a$$

$$y = -x + 3a$$ //

TRUTH: Righteousness keepeth him that is upright in the way but wickedness overthroweth the sinner (Proverb 13 : 6)

SOLVED PAST QUESTIONS AND ANSWERS

2011: No 29: The coordinates of vertex of the parabola $x = 2y^2 - 4y + 5$ is (A) (1, 3) (B) (3, 1) (C) (-1, 3) (D) (1, -3) (E) (-1, -3)

SOLUTION

$x = 2y^2 - 4y + 5$

$2y^2 - 4y = x - 5$

$2(y^2 - 2y) = x - 5$

Find half of the coefficient of y, square it, add it to L.H.S and subtract it again

$2(y^2 - 2y + 1^2 - 1^2) = x - 5$

$2(y^2 - 2y + 1 - 1) = x - 5$

$2y^2 - 4y + 2 - 2 = x - 5$

$2y^2 - 4y + 2 = x - 5 + 2$

$2(y^2 - 2y + 1) = x - 3$

$2(y-1)^2 = x - 3$

$(y-1)^2 = \frac{1}{2}(x-3)$ (5)

Vertex $= (0, 0) = (\bar{X}, \bar{Y})$

$\bar{X} = x - x_0 = 0 \qquad \bar{Y} = y - y_0 = 0$

$y - 1 = 0$ and $x - 3 = 0$

$y = 0 + 1$ and $x = 0 + 3$

$y = 1 \qquad x = 3$ (B)

$(x, y) = (3, 1)$

(31) The parabola with focus (1, 1) and directrix $y = -1$ is

(A) $y = -\frac{1}{4}(x-1)^2$ (B) $y = -\frac{1}{4}(x+1)^2$ (C) $y = \frac{1}{4}(x-1)^2$ (D) $x = -\frac{1}{4}(y-1)^2$ (E) $x = \frac{1}{4}(y-1)^2$

SOLUTION

directrix is $y = -1$

$y + 1 = 0$

distance of directrix $= |directrix| = y + 1$

Focus $= (1, 1)$

distance of focus $= |Focus| = \sqrt{(x-x_1)^2 + (y-y_1)^2}$

$|Focus| = \sqrt{(x-1)^2 + (y-1)^2}$

$|Focus| = |directrix|$ for parabola

$y + 1 = \sqrt{(x-1)^2 + (y-1)^2}$

square both side

$(y+1)^2 = (x-1)^2 + (y-1)^2$

$\cancel{y^2} + 2y + 1 = x^2 - 2x + 1 + \cancel{y^2} - 2y + 1$

$x^2 - 2x + 1 = 2y + 2y + \cancel{1} - \cancel{1}$

$(x-1)^2 = 4y$ or (C)

$y = \frac{1}{4}(x-1)^2$

2009: No 1: The equation with the directrix $x = -3$ and focus (3, 0) is (A) $x^2 = 6y$ (B) $y^2 = 12x$ (C) $y^2 = 6x$ (D) $x^2 = -6y$ (E) None of the above

SOLUTION

directrix $x = -3$ or $x + 3 = 0$

$|directrix| = |Focus| \qquad Focus = (3, 0)$

$x + 3 = \sqrt{(x-3)^2 + (y-0)^2}$

Square both side

$(x+3)^2 = (x-3)^2 + y^2$

$\cancel{x^2} + \cancel{9} + 6x = \cancel{x^2} + \cancel{9} - 6x + y^2$

$y^2 = 6x + 6x$ (B)

$y^2 = 12x$

(7) The length of the lactum rectum of the conic described by $x^2 - 6x - 2y + 5 = 0$ is (A) -2 (B) -3 (C) 1 (D) 2 (E) 4

SOLUTION

$x^2 - 6x - 2y + 5 = 0$

$x^2 - 6x = 2y - 5$

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$x^2-6x+3^2-3^2=2y-5$

$x^2-6x+9=2y-5+9$

$(x-3)^2=2y+4$

$(x-3)^2=2(y+2)$ — derived

$x^2=4ay$ --- standard

$4a=2 \quad a=2/4=1/2$

Length of Lactum rectum $=4a=4\times\frac{1}{2}$
$=2$

D

Aliter: Length of Lactum rectum is the absolute value of the coefficient of y (i.e $-2y$) which is 2//

4) The axes of the parabola $x^2=4ay$ is (A) $y=0$ (B) $x=0$ (C) $x=a$ (D) $y=a$ (E) $x=1$

SOLUTION

D

2012: No 40: The line of symmetry of the parabola $x^2=-4ay$ is
(A) +ve y-axes (B) -ve y-axes
(C) +ve x-axes (D) -ve x-axes (E) None

SOLUTION

B

6

2009: No 5: Which of the following equation does not represent a conic? (A) $x^2-2y^2+5x=7$
(B) $5x^2+5y^2+4x-10x-10y-2=0$
(C) $6y^2-2x+2y+4=0$
(D) $3x^2+4y^2-8x+6=0$
(E) $3x^2-3y^2+2x-3y=0$

SOLUTION

For hyperbola, coefficient of x^2 and y^2 are not equal and second(y), they have opposite sign (+ and −) e.g
$x^2-2y^2+5x=7$

For circle, the coefficient of x^2 and y^2 are equal and they are of the same sign e.g
$5x^2+5y^2+4x-10y-2=0$

For ellipse the coefficient of x^2 and y^2 are not equal but they are of the same sign e.g $3x^2+4y^2-8x+6=0$

For parabola, the highest power of x and y are not equal e.g if x is to power 2, y is to power 1 e.g
$6y^2-2x+2y+4=0$

NOTE: the highest power of x and y are the same for the rest (x^2 and y^2)

For rectangular parabola, the coefficient of x^2 and y^2 are numerically equal but of opposite sign e.g
$3x^2-3y^2+2x-3y=0$

B

Similarity: None of them has xy-terms
Circle is NOT a conic section

26) The eccentricities of three conic sections are given as $\frac{4}{3}, 1, \frac{9}{10}$ are respectively (A) parabola, Ellipse, Hyperbola (B) Ellipse, Parabola, Hyperbola (C) Ellipse, hyperbola, parabola (D) Hyperbola, parabola, ellipse (E) None

SOLUTION

Hyperbola $e=4/3$ $(e>1)$

Parabola $e=1$ $(e=1)$

D

ellipse $e=9/10$ $(e<1)$

NOTE For circle $e=0$ but circle is NOT a conic section

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TRUTH: Only by pride cometh contention but with the well-advised is wisdom (Proverb 13:10)

SOLVED PAST QUESTIONS AND ANSWERS

2013 NO12: The length of the lactum rectum of the parabola $y^2+12y+8x+20=0$ is (A) -8 (B) 8 (C) 4 (D) -4 (E) 2

SOLUTION

$$y^2+12y+8x+20=0$$

$$y^2+12y=-8x-20$$

Find half of the coefficient of y, square it, add it to LHS and subtract it again

$$y^2+12y+6^2-6^2=-8x-20$$

$$y^2+12y+36=-8x-20+36$$

$$(y+6)^2=-8x+16$$

B

$(y+6)^2=-8(x-2)$ -- derived

Compare with $y^2=-4ax$ --- standard

$-4a=-8 \quad a=8/4=2$

Length of Lactum rectum $=4a=4\times2=8$

Alliter: length of lactum rectum is absolute value of the coefficient of x (i.e $-8x$) in the equation, L R $=8$

7

14) The end points of the lactum rectum of the parabola $(y-1)^2=4x-8$ are at (A) (1, 3) and (3, -1) (B) (3, 3) and (3, -1) (C) (2, 0) and (-2, 0) (D) (0, 2) and (0, -2) (E) (3, 1) and (3, -1)

SOLUTION

$(y-1)^2=4x-8$

$(y-1)^2=4(x-2)$ derived

Compare with $y^2=4ax$ standard

$4a=4$

$a=1$

For $y^2=4ax$, coordinate of lactum rectum $=(a, \pm2a)=(\bar{X}, \bar{Y})$

$\bar{X}=x-x_0=a \quad \bar{Y}=y-y_0=\pm2a$

$x-2=a$ and $y-1=\pm2a$

$x=2+a$ and $y=1\pm2a$

$x=2+1$ and $y=1+2(1)$ or $1-2(1)$

$x=3$ and $y=3$ or -1

B

end point of the lactum rectum $=(x, y)=(3,3)$ & $(3,-1)$

18) The equation of a normal to the parabola $y^2=4ax$ at $(1,0)$ is given as (A) $x=0$ (B) $2y=a$ (C) $2y+ax+1=0$ (D) $y=0$ (E) $y+1=1$

SOLUTION

$y^2=4ax$

differentiate with respect to x

$2y\frac{dy}{dx}=4a$ at $(1,0)$

$\frac{dy}{dx}=\frac{4a}{2y}=\frac{4a}{2(0)}=\frac{4a}{0}$

equation of normal is

$y-y_1=\frac{-1}{m}(x-x_1)$

$y-0=\frac{-1}{4a/0}(x-1)$

D

$y-0=\frac{-1\times0}{4a}(x-1)$

$y-0=0 \quad y=0$

H) The equation of tangent to the parabola $y^2=4ax$ at the point $(at^2, 2at)$ is

(A) $x+ty+at^2=0$ (B) $x+ty=at^2$ (C) $y-tx=at^2$ (D) $ty=x+at^2$ (E) $y+tx=at^2$

SOLUTION

$y^2 = 4ax$ at $(at^2, 2at)$

$2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{4a}{2y} = \frac{4a}{2(2at)} = \frac{1}{t}$

$y - y_1 = m(x - x_1)$

$y - 2at = \frac{1}{t}(x - at^2)$

$t(y - 2at) = x - at^2$

$ty - 2at^2 = x - at^2$

$ty = x - at^2 + 2at^2$

$ty = x + at^2$

D

(25) The vertex of the parabola is $4(x+5)^2 - 3y + 2 = 0$ is (A) $(-5, 3/2)$ (B) $(0, 0)$ (C) $(-5, 2/3)$ (D) $(-1, 3)$ (E) $(-5, 1)$

SOLUTION

$4(x+5)^2 - 3y + 2 = 0$

$4(x+5)^2 = 3y - 2$

factorize 3 from R.H.S

$4(x+5)^2 = 3(y - \frac{2}{3})$

$(x+5)^2 = \frac{3}{4}(y - 2/3)$

vertex is $(0, 0) = (\bar{X}, \bar{Y})$ i.e

$\bar{X} = x - x_0 = 0$, $\bar{Y} = y - y_0 = 0$

$x + 5 = 0$ and $y - 2/3 = 0$

$x = 0 - 5$ and $y = 0 + 2/3$

$x = -5$ and $y = 2/3$

$(x, y) = (-5, 2/3)$

C

2012: No 39. The slope of the normal to the parabola $y^2 = 4ax$ at the point $P(at^2, 2at)$ is (A) $\frac{1}{t}$ (B) $-t$ (C) a/t (D) $-1/t^2$ (E) t.

SOLUTION

$y^2 = 4ax$ at $(at^2, 2at)$

$2y\frac{dy}{dx} = 4a$

$m = \frac{dy}{dx} = \frac{4a}{2y} = \frac{4a}{2(2at)} = \frac{1}{t}$

slope of normal $m_1 = \frac{-1}{m} = \frac{-1}{1/t} = -t$

B

2011: No 13: The focus of the parabola $y^2 = -40x$ is (A) $(10, 0)$ (B) $(0, 10)$ (C) $(-10, 0)$ (D) $(0, -10)$ (E) $(-10, 10)$

SOLUTION

$y^2 = -40x$ --- given

Compare with $y^2 = -4ax$ -- standard

$-4a = -40$ $a = \frac{-40}{-4} = 10$

Focus $= (-a, 0) = (-10, 0)$

C

19) The equation of the directrix of the parabola $y^2 = 32x$ is (A) $x = -8$ (B) $y = -8$ (C) $x = 9$ (D) $y = 8$ (E) $x = -9$

SOLUTION

$y^2 = 32x$ -- given

$y^2 = 4ax$ -- standard

$4a = 32$ $a = 32/4 = 8$

equation of directrix is $x = -a$

$x = -8$

A

(32) The directrix of the parabola $y^2 = -40x$ is (A) $y = 10$ (B) $y = -10$ (C) $x = -40$ (D) $x = -10$ (E) $x = 10$

SOLUTION

$y^2 = -40x$ -- given

$y^2 = -4ax$ --- standard

$-4a = -40$

$a = \frac{-40}{-4} = 10$

equation of directrix is $x = a$

$x = 10$

E

TRUTH: Wealth gotten by Vanity shall be diminished but he that gathereth by labor shall increase (Proverb 13:11)

PAST QUESTIONS AND ANSWERS

2009 No17: The straight line $x = 3$ intercept the parabola $y^2 = 4(x+1)$ at (A) $(3,-4)$ and $(3,4)$ (B) $(3,4)$ and $(4,3)$ (C) $3, 16$ and $(3,-16)$ (D) $(3,-3)$ and $(3,3)$ (E) None

Solution

you monitor me, you die

$x = 3$ ---- eqn (i)

$y^2 = 4(x+1)$ --- eq (ii) (A)

Substitute for x in eqn (ii)

$y^2 = 4(x+1) = 4(3+1) = 4 \times 4 = 16$

$y = \sqrt{16} = \pm 4 = +4$ or -4

Intercept $= (x, y) = (3, 4)$ and $(3, -4)$

2008: No36: The equation of the parabola with the directrix $x = 0$ and focus $(6,1)$ is (A) $x^2 - 3x - 12y + 27 = 0$ (B) $x^2 - 2x + 12y + 27 = 0$ (C) $y^2 - 2y - 12x + 37 = 0$ (E) $y^2 - 3y - 12x + 10 = 0$ (E) None

SOLUTION

$|\text{directrix}| = |\text{Focus}|$ 9

$|x - 0| = \sqrt{(x-6)^2 + (y-1)^2}$

Square both side (C)

$(x-0)^2 = (x-6)^2 + (y-1)^2$

$\cancel{x^2} = \cancel{x^2} + 36 - 12x + y^2 - 2y + 1$

$y^2 - 2y - 12x + 37 = 0$

TUTORIAL QUESTIONS

A tangent line to the parabola $3x + y^2 = 12$ is parallel to the line $y = 2x + k$ find (i) equation of the line tangent to parabola (ii) value of k

SOLUTION

$y = 2x + k \qquad c = k$

$y = mx + c \qquad m = 2$

$3x + y^2 = 12$

$y^2 = -3x + 12$

$y^2 = -3(x - 4)$

$y^2 = -4ax$

$-4a = -3 \quad a = 3/4$

point of contact of tangent to parabola is

$\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{3/4}{2^2}, \frac{\cancel{2}(3/4)}{\cancel{2}}\right) = \left(\frac{3}{16}, \frac{3}{4}\right)$

Equation of tangent to the parabola

$y - y_1 = m(x - x_1)$

$y - 3/4 = 2(x - 3/16)$ multiply through by 8

$8y - 6 = 16x - 3$

$8y - 16x - 3 = 0$

1) Find the equation of the tangent and normal to the parabola $y^2 = 12x$ at point where ordinate is 6

SOLUTION

ordinate $\rightarrow$ y - axis

abscissa $\rightarrow$ x - axis

$y = 6$ --- eq (i)

$y^2 = 12x$ --- eq (ii)

substitute for y in eq (ii)

$y^2 = 12x$ or $6^2 = 12x \quad x = 36/12 = 3$

$y^2 = 12x$ at $(3, 6)$

differentiate with respect to x

$2y\frac{dy}{dx} = 12$ or $\frac{dy}{dx} = 12/2y$

$m = \frac{dy}{dx} = \frac{12}{2(6)} = 1$

Equation of tangent to the parabola

$y - y_1 = m(x - x_1)$

$y - 6 = 1(x - 3) \qquad \boxed{y - x - 3 = 0}$

Equation of normal to the parabola

$y - y_1 = \frac{-1}{m}(x - x_1)$

$y - 6 = \frac{-1}{1}(x - 3)$

$y - 6 = -x + 3 \qquad \boxed{y + x - 9 = 0}$

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TRUTH: Whosoever despiseth the word shall be destroyed but he that feareth the Commandment shall be rewarded (Proverbs 13:13)

PARAMETRIC EQUATION OF PARABOLA.

The parametric equation of the parabola $y^2 = 4ax$ is at point $x = at^2$ and $y = 2at$ i.e $(x, y) = (at^2, 2at)$

Equation of tangent to the parabola (which has a slope of $m = 1/t$) is

$$\boxed{yt = x + at^2}$$

Equation of normal to the parabola (which has a slope of $m = -t$) is

$$\boxed{y = at^3 + 2at - xt}$$

point of intersection of two tangent line on parabola is given as

$(x, y) = [at_1t_2, a(t_1 + t_2)]$

EXAMPLE

Prove that $x = 3t^2 + 1$ and $y = \frac{1}{2}(3t+1)$ are the parametric equation of a parabola. Find its vertex and length of lactum rectum

Solution

$x = 3t^2 + 1$, $y = 1/2(3t+1)$

$x - 1 = 3t^2$ $\quad$ $2y = 3t + 1$

$t^2 = \frac{x-1}{3}$ $\quad$ $2y - 1 = 3t$

$t = \sqrt{\frac{x-1}{3}}$ $\quad$ $t = \frac{2y-1}{3}$

$t = t$

$$\sqrt{\frac{x-1}{3}} = \frac{2y-1}{3}$$

Square both side

$$\frac{x-1}{3} = \left(\frac{2y-1}{3}\right)^2$$

$$\frac{x-1}{3} = \frac{(2y-1)^2}{3^2}$$

$$\frac{x-1}{3} = \frac{(2y-1)^2}{9}$$

$$(2y-1)^2 = \frac{9}{3}(x-1)$$

$$(2y-1)^2 = 3(x-1)$$

$$4y^2 - 4y + 1 = 3x - 3$$

$$4y^2 - 4y = 3x - 3 - 1$$

$$4(y^2 - y) = 3x - 4$$

Find half of the coefficent of y, square it, add it to L.H.S and subtract it again

$$4\left(y^2 - y + \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2\right) = 3x - 4$$

$$4\left(y^2 - y + \tfrac{1}{4} - \tfrac{1}{4}\right) = 3x - 4$$

$$4y^2 - 4y + 1 - 1 = 3x - 4$$

$$4y^2 - 4y + 1 = 3x - 4 + 1$$

$$4(y^2 - y + 1/4) = 3x - 3$$

$$y^2 - y + 1/4 = \frac{3}{4}(x-1)$$

$$(y - 1/2)^2 = \frac{3}{4}(x-1)$$

Vertex $= (0, 0) = \bar{X}, \bar{Y})$

$y - 1/2 = 0$ and $x - 1 = 0$

$(x, y) = (1, 1/2)$

$$(y - 1/2)^2 = 3/4(x-1)$$

$$y^2 = 4ax$$

Length of Lactum rectum $= 4a = 3/4$ //

EXAMPLE

Find the focus and directrise of $(3 - 2t^2, 4t)$

SOLUTION

$x = 3 - 2t^2$, $\quad$ $y = 4t$

$2t^2 = 3 - x$ $\quad$ $4t = y$

$t^2 = \frac{3-x}{2}$ $\quad$ $t = y/4$

$t = \sqrt{\frac{3-x}{2}}$ $\quad$ $t = y/4$

$t = t$

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$\sqrt{\frac{2-x}{2}} = \frac{y}{4}$

Square both side

$\frac{2-x}{2} = \frac{y^2}{16}$

$y^2 = 8(2-x)$

$y^2 = 8(2-x)$

$y^2 = 8(-x+2)$

$y^2 = -8(x-2)$

$y^2 = -4ax$

$-4a = -8$ $a = 2$

Equation of directrix $X = a$

$x - 2 = 2$ $x = 2 + 2 = 4$

Focus $= (-a, 0)$

SOLVED PAST QUESTIONS

2015: No 14: Given that $x = \sqrt{t}$, $y = \sqrt{9-t}$ eliminating the parameter t, the equation of the conic is
(A) $x^2 - y^2 = 9$ (B) $x^2 + 9y^2 = 9$ (C) $x^2 + y^2 = 9$
(D) $x^2 + 4y^2 = 9$ (E) None

SOLUTION

$x = \sqrt{t}$

Square both side

$x^2 = t$

$t = x^2$

$y = \sqrt{9-t}$

Square both side

$y^2 = 9 - t$

$t = 9 - y^2$

$t = t$

$x^2 = 9 - y^2$ (C)

$x^2 + y^2 = 9$

2019 No 31: The vertex of the curve described by the equations $x = 2t^2 - 1$, $y = 3t$ is (A) $(0, 2)$ B $(-1, 0)$ (C) $(0, -1)$ (D) $(0, 1)$ (E) None

SOLUTION

$x = 2t^2 - 1$

$2t^2 = x + 1$

$t^2 = \frac{x+1}{2}$

$t = \sqrt{\frac{x+1}{2}}$

$y = 3t$

$3t = y$

$t = y/3$

$t = y/3$

12

$t = t$

$\sqrt{\frac{x+1}{2}} = \frac{y}{3}$

Square both side

$\frac{x+1}{2} = \frac{y^2}{9}$

$y^2 = \frac{9}{2}(x+1)$ (B)

Vertex $= (0, 0) = (X, Y)$

$x + 1 = 0$ and $y = 0$

$x = -1$ and $y = 0$ $(x, y) = (-1, 0)$

2001/2002 NUMBER 16 (C.A TEST)

Find the equation of the tangent to the vertex of the parabola $x = 2t - 1$, $y = t^2 + 2$

SOLUTION

$x = 2t - 1$

$2t = x + 1$

$t = \frac{x+1}{2}$

$y = t^2 + 2$

$t^2 = y - 2$

$t = \sqrt{y-2}$

$t = t$

$\frac{x+1}{2} = \frac{\sqrt{y-2}}{1}$

Square both side

$\frac{(x+1)^2}{4} = \frac{y-2}{1}$

$(x+1)^2 = 4(y-2)$

Vertex $(0, 0) = (X, Y)$

$X = x + 1 = 0$ and $Y = y - 2 = 0$

$x = -1$ and $y = 2$

From the value above

$x = -1$ and $x = 2t - 1$

$2t - 1 = -1$ or $2t = -1 + 1$

$2t = 0$ $t = 0$

Equation of parametric tangent to parabola is given as

$yt = x + at^2$ when $t = 0$

$y(0) = x + 9(0)^2$

$0 = x + 0$

$x = 0$

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TRUTH: Poverty and shame shall be to him that refuseth instruction (Proverb 13:18)

PAST QUESTIONS AND ANSWERS

2015 NO 2: The equation of the tangent to the curve $x^2 = 4y$ at the point $(2,-1)$ is (A) $y+x=1$ (B) $y-x=-3$ (C) $y-x=-1$ (D) $x-3=3$ (E) NONE

SOLUTION

$$x^2 = 4y$$

differentiate with respect to x

$$2x = 4\frac{dy}{dx} \quad \text{at } (2,-1)$$

$$m = \frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2} = \frac{2}{2} = 1$$

Equation of tangent to the parabola at $(2,-1)$ is given as $y - y_1 = m(x - x_1)$

$$y - -1 = 1(x-2)$$

$$y+1 = x-2$$

$$y = x-2-1$$

$$y = x-3 \quad \text{or} \quad y - x = -3$$

D

NO 3: The equation of a parabola when the focus is at $(3,0)$ and vertex $(1,0)$ is given as (A) $y^2 = 8(x+1)$ (B) $(y-1)^2 = 8x$ (C) $y^2 = 8(x-1)$ (D) $y^2 = 12(x-1)$ (E) NONE

13

SOLUTION

Given focus $= (x_1, y_1) = (3,0)$ and vertex $= (\bar{x}, \bar{y})$. If the coordinate point is (x_2, y_2). The vertex is the midpoint between focus and coordinate point

$$\bar{x}, \bar{y} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$(1,0) = \left(\frac{3+x}{2}, \frac{0+y}{2}\right)$$

$$\frac{3+x}{2} = 1 \qquad \frac{0+y}{2} = 0$$

$$3+x = 2 \qquad 0+y = 0$$

$$x = 2-3 \qquad y = 0$$

$$x = -1 \qquad y' = 0$$

Coordinate point $(x, y) = (-1, 0)$

Focus of parabola $(x, y) = (3, 0)$

The directrix will be gotten from any cartesian point in the coordinate point that is different from that of the focus i.e directrix is $x = -1$ or $x+1=0$ since the focus is given as $(3,0)$

distance from focus $= |\text{Focus}| = \sqrt{(x-x_1)^2 + (y-y_1)^2}$

$$|\text{Focus}| = \sqrt{(x-3)^2 + (y-0)^2}$$

$$|\text{Focus}| = \sqrt{x^2 + 9 - 6x + y^2}$$

distance from directrix $= |\text{directrix}| = x+1$

For parabola $|\text{directrix}| = |\text{Focus}|$

$$x+1 = \sqrt{x^2+9-6x+y^2}$$

Square both side

$$(x+1)^2 = x^2+9-6x+y^2$$

$$\cancel{x^2}+1+2x = \cancel{x^2}+9-6x+y^2$$

$$y^2 = 1+2x+6x-9$$

$$y^2 = 8x+1-9$$

$$y^2 = 8x+8 \qquad y^2 = 8(x+1)$$

A

5) The length of lactum rectum of the curve $y^2 - 8y - 16x + 48 = 0$ is (A) 1 (B) 10 (C) 4 (D) 16 (E) None

SOLUTION

$$y^2 - 8y - 16x + 48 = 0$$

$$y^2 - 8y = 16x - 48$$

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find half of the coefficient of y, square it, add it to L.H.S and subtract it again

$y^2 - 8y + 4^2 - 4^2 = 16x - 48$

$y^2 - 8y + 16 = 16x - 48 + 4^2$

$(y-4)^2 = 16x - 48 + 16$

$(y-4)^2 = 16x - 32$ or $(y-4)^2 = 16(x-2)$

$(y-4)^2 = 16(x-2) \rightarrow$ derived

Compare with $y^2 = 4ax \rightarrow$ standard

$4a = 16$ (D)

Length of Lactum rectum $= 4a = 16$

Allter: Length of Lactum is absolute [redacted] value of coefficient of x in the given equation i.e $-16x$ L.R $= 16$

16) A parabola with length of Lactum rectum $\frac{1}{4}$ and vertex $(0,1)$ is given as (A) $4(y-1)^2 = x$ (B) $4(x-1)^2 = y$ (C) $4(y+1)^2 = x$ (D) $(y-1)^2 = 4(x-1)$ (E) None

SOLUTION

Vertex $= (0, 1)$ $\quad x = 0$

Given ~~point~~ is $y = 1$ or $y - 1 = 0$

$L.R = 4a = 1/4$

$$\boxed{(y-y_1)^2 = 4a(x-x_1)} \quad /14$$

$(y-1)^2 = \frac{1}{4}(x-0)$

$4(y-1)^2 = x$ (A)

23) If the Lactum rectum of a parabola is $3/\sqrt{5}$, its focal Length is (A) $\frac{3}{2\sqrt{5}}$ (B) $\frac{3}{\sqrt{5}}$ (C) $\frac{3}{4\sqrt{5}}$ (D) $\frac{3}{4}$ (E) None

SOLUTION

Length of lactum rectum $= 4a = 3/\sqrt{5}$

focal Length $= a = \frac{3}{4\sqrt{5}} \Rightarrow$ (C)

28) The equation of tangent to the parabola $x^2 = 4ay$ at the point $P(2a, -1)$ is (A) $y = x + 2a - 1$ (B) $y = x - (2a+1)$ (C) $y = x - 2a + 1$ (D) $y = x + 2a + 1$ (E) None

SOLUTION

$x^2 = 4ay$ at $(2a, -1)$

diffrentiate with respect to x

$2x = 4a\frac{dy}{dx}$

$m = \frac{dy}{dx} = \frac{2x}{4a} = \frac{2(2a)}{4a} = 1$

$y - y_1 = m(x - x_1)$

$y - -1 = 1(x - 2a)$ (B)

$y + 1 = x - 2a$

$y = x - 2a - 1$ or

$y = x - (2a+1)$

2014: No 6: Equation of tangent to the ~~to~~ the parabola $x^2 = -6y$ which is perpendicular to the Line $y - x = 2$ (A) $y + x = 2$ (B) $y - x = 2$ (C) $y = x - \frac{3}{2}$ (D) $y = x + \frac{3}{2}$ (E) $y = -x - \frac{3}{2}$

SOLUTION

$y - x = 2$

$y = x + 2 \quad m = 1$

$y = mx + c \quad c = 2$

The tangent is perpendicular to the line i.e $m_1 = \frac{-1}{m} = \frac{-1}{1} = -1$

Compare with

$x^2 = -6y \rightarrow$ given (E)

$x^2 = 4ay \rightarrow$ standard

$4a = -6$ or $a = -3/2$

Equation of tangent $y = mx + \frac{a}{m}$

$y = (-1)x + \frac{-3/2}{(-1)}$ $\quad y = -x - \frac{3}{2}$

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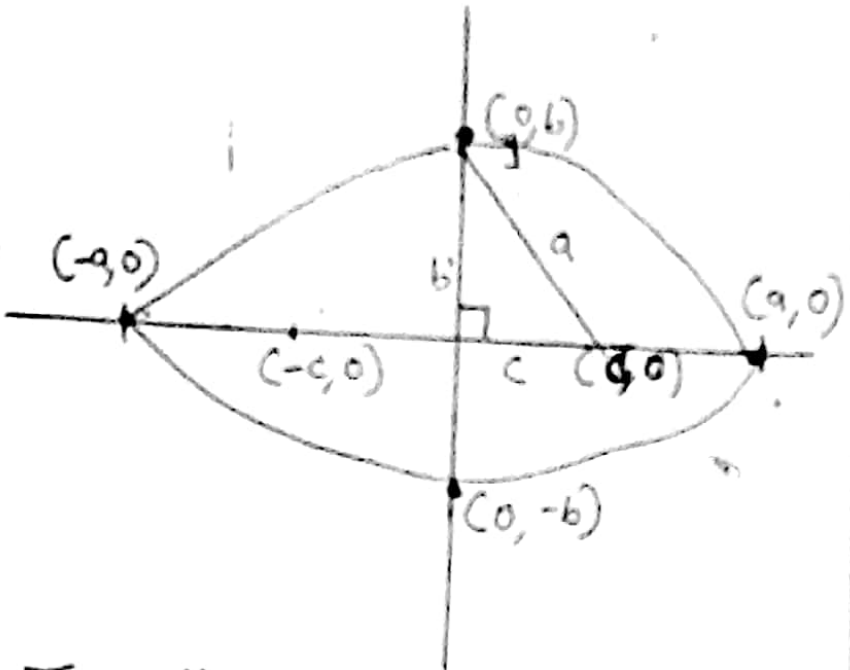
TRUTH: Without counsel purposes are disappointed but in the multitude of Counselors they are established (Proverb15:22)

ELLIPSE

There are two categories of ellipse

TYPE A



This ellipse has its equation as with respect to origin (0,0) given

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a \geq b,\ b > 0$$

Its foci $= (\pm c, 0)$ i.e which are $(+c, 0)$ and $(-c, 0)$ where

$$c = \sqrt{a^2-b^2} \text{ or } c = ae$$

Its vertices $= (\pm a, 0)$ i.e which are $(+a, 0)$ and $(-a, 0)$

Its Convertices $= (0, \pm b)$ i.e which are $(0, +b)$ and $(0, -b)$

NOTE: $(\pm a, 0)$ and $(0, \pm b)$ are coordinate at the end of major axis (intercept of x) and minor axis (intercept of y) respectively. They are called vertices and convertices respectively

The point of intersection of major and minor axis which is known as the centre has its coordinate $= (0, 0)$

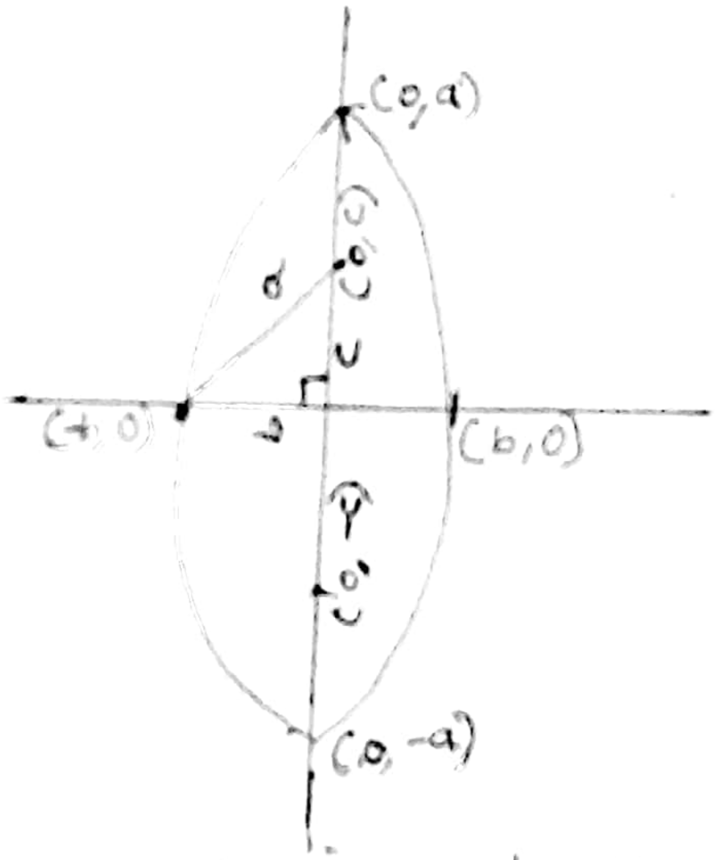
equation of major axis is given as $y = 0$

equation of minor axis is given as $x = 0$

Equation of directrix is given as $x = \pm a/e$

Coordinate of lactum Rectum is $= (\pm c, b^2/a)$ i.e which are $(+c, b^2/a)$ and $(-c, b^2/a)$ where $c = \sqrt{a^2-b^2}$ or $c = ae$

TYPE B



This ellipse has its equation as with respect to origin (0,0) given

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a \geq b,\ b > 0$$

Its foci $= (0, \pm c)$ i.e which are $(0, +c)$ and $(0, -c)$ where

$$c = \sqrt{a^2-b^2} \text{ or } c = ae$$

Its vertices $= (0, \pm a)$ i.e which are $(0, +a)$ and $(0, -a)$

Its Convertices $= (\pm b, 0)$ i.e

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which are $(+b, 0)$ and $(-b, 0)$

Coordinate of point of interception $= (0, 0)$

equation of major axis is given as

$$x = 0$$

equation of minor axis is given as

$$y = 0$$

equation of directrix is given as

$$y = \pm a/e$$

Coordinate of lactum rectum is given as $(c, \pm b^2/a)$ which are $(c, +b^2/a)$ and $(c, -b^2/a)$ as $c = \sqrt{a^2-b^2} = ae$

GENERAL PROPERTIES OF ELLIPSE

Length of major axis $= 2a$

Length of semi-major axis $= a$

Length of minor axis $= 2b$

Length of semi minor axis $= b$

Length of lactum rectum $= 2b^2/a$

QUADRANT OF COORDINATE OF LACTUM RECTUM OF ELLIPSE

2nd quadrant	1st quadrant
$(-ae, b^2/a)$	$(+ae, b^2/a)$
3rd quadrant	4th quadrant
$(ae, -b^2/a)$	$(ae, +b^2/a)$

16

EQUATION OF ELLIPSE WITH THE ECCENTRICITY (e) CENTRE

The ratio of the distance from fixed line (directrix) to fixed point (focus) is called eccentricity (e). Mathematically

$$e = \frac{|Focus|}{|directrix|} \quad \text{or} \quad \frac{1}{e} = \frac{|directrix|}{|Focus|}$$

For ellipse $e < 1$ e.g

If a point moves so that its distance from (a, b) is half its distance from the line $y = mx + c$ then the eccentricity $(e) = 1/2$. If

$$e = \frac{|Focus|}{|directrix|} \quad \boxed{e < 1}$$

$$|Focus| = e\,|directrix|$$

Given that the focus is (x, y) with respect to given point (x_1, y_1)

The distance of the focus is

$$|Focus| = \sqrt{(x-x_1)^2 + (y-y_1)^2}$$

Also the fixed line (directrix) is perpendicular to the axis at point (x_1, y_1) such that the directrix $y = mx + c$ or $ax + by + c = 0$ has its distance as

$$|directrix| = \frac{ax + by + c}{\sqrt{a^2 + b^2}}$$

Also $b^2 = a^2(1 - e^2)$

Locus of point of perpendicular tangent to the ellipse is given as $x^2 + y^2 = a^2 - b^2$

Equation of tangent to ellipse is given as

$$\frac{yy_1}{b^2} + \frac{xx_1}{a^2} = 1$$

Equation of Normal to ellipse is given as

$$\frac{y - y_1}{y_1} = \frac{a^2}{b^2}\frac{(x - x_1)}{x_1}$$

These last two equation is for type A ellipse

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TRUTH: The Lord is far from the wicked but he heareth the ~~prayer of the~~ righteous (Proverb 15: 29)

EXAMPLE

An ellipse of eccentricity $2/3$ has the point $(3,2)$ and $(7,2)$ as foci. Find

I) the length of lactum rectum

II) Coordinate of its centre

III) equation of the ellipse

IV) equation of directrix

SOLUTION

Foci $= (3,2)$ and $(7,2)$

NOTE: Since the foci is not $(\pm c, 0)$ then it mean the ellipse is not from the origin $(0,0)$. Then the foci is $(h \pm c, k)$ where $c = ae$

foci $= (h+ae, k)$ and $(h-ae, k)$

Foci $= (7, 2)$ and $(3, 2)$

Therefore $h + ae = 7$ ---- eqn (I)

$h - ae = 3$ ---- eqn (II)

add eqn (I) to eq (II)

$2h = 10 \quad h = 10/2 = 5$

$h + ae = 7, \; 5 + ae = 7$

$\boxed{ae = 2}$ where $e = 2/3$

$a = 2/e = 2/{2/3} = 3$ If $c = ae = 2$

and $a = 3$ then $c = \sqrt{a^2 - b^2}$

$c^2 = a^2 - b^2$ or $b^2 = a^2 - c^2$

$b^2 = 3^2 - 2^2 \Rightarrow b = \sqrt{9-4} = \sqrt{5}$

I) Length of lactum rectum $= \frac{2b^2}{a}$

$L \cdot R = \frac{2 \times (\sqrt{5})^2}{3} = 10/3$

II) Coordinate of its centre is the mid point of the foci

$(\bar{x}, \bar{y}) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$(\bar{x}, \bar{y}) = \left(\frac{3+7}{2}, \frac{2+2}{2}\right) = (5, 2)$

Equation of ellipse with respect to the centre of coordinate $(5, 2)$

$$\frac{(x-\bar{x})^2}{a^2} + \frac{(y-\bar{y})^2}{b^2} = 1$$

$$\Rightarrow \frac{(x-5)^2}{3^2} + \frac{(y-2)^2}{(\sqrt{5})^2} = 1$$

$$\Rightarrow \frac{(x-5)^2}{9} + \frac{(y-2)^2}{5} = 1$$

NOTE: Equation of directrix with respect to origin $(0,0)$ is $x = \pm a/e$ but the equation of directrix with respect to coordinate of the centre is given as

$\bar{X} = x - \bar{x} = \pm a/e$

If the ellipse is of type B, then the equation of directrix with respect to centre of coordinate is given as

$\bar{Y} = y - \bar{y} = \pm a/e$

Therefore, $x - 5 = \pm 3/{2/3}$

$x - 5 = \pm 9/2 \quad x = 5 \pm 9/2$

$x = 5 + \frac{9}{2}$ or $x = 5 - 9/2$

$x = 19/2$ or $x = 1/2$

$x = 9\frac{1}{2}$ or $x = 1/2$

SOLVED PAST QUESTIONS AND ANSWERS

2015 No11: The eccentricity e of the Conics $\frac{(x+1)^2}{5} + \frac{(y-5)^2}{5} = 1$ is

(A) $e = 1$ (B) $e < 1$ (C) $e > 1$ (D) $e = 0$ (E) None

SOLUTION

$$\frac{(x+1)^2}{5} + \frac{(y-5)^2}{5} = 1$$

$$\frac{(x-\bar{x})^2}{a^2} + \frac{(y-\bar{y})^2}{b^2} = 1$$

$a^2 = 5, \quad b^2 = 5$ where

$b^2 = a^2(1 - e^2)$

$1 - e^2 = b^2/a^2 \qquad 1 - e^2 = 5/5$

$1 - e^2 = 1$ or $e^2 = 1 - 1 \quad e = 0$ (D)

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2014: No 1 The ellipse with centre at origin, major axis as 10 and foci as $(\pm 4, 0)$ has equation (A) $\frac{x^2}{9} - \frac{y^2}{25} = 1$ (B) $\frac{x^2}{9} + \frac{y^2}{25} = 1$ (C) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ (D) $\frac{x^2}{25} + \frac{y^2}{36} = 1$ (E) $\frac{x^2}{36} + \frac{y^2}{25} = 1$

SOLUTION

Length of major axis $= 2a = 10$

$a = 10/2 = 5$

Foci $= (\pm c, 0) = (\pm 4, 0)$ $c = 4$

$c^2 = a^2 - b^2$ or $b^2 = a^2 - c^2$ then

$b^2 = 5^2 - 4^2$ $b^2 = 25 - 16$ $b^2 = 9$

$b^2 = 9$, $b = \sqrt{9} = 3$

NOTE: The foci given in the question determine wether the ellipse will be type A or type B e.g

for $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ foci $= (\pm c, 0)$

for $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ foci $= (0, \pm c)$

Can you see that different foci have different equation? O.K

for $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ foci $= (\pm c, 0)$

$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ foci $= (\pm 4, 0)$

$\frac{x^2}{25} + \frac{y^2}{9} = 1$ (C)

18

(18) The foci of the ellipse $x^2 + 2y^2 - 6x + 4y + 7 = 0$ are

(A) $(1 \pm \sqrt{2}, -2)$ (B) $(2 \pm \sqrt{2}, -1)$ (C) $(3 \pm \sqrt{2}, -1)$ (D) $(1 \pm \sqrt{2}, -1)$ (E) $(3 \pm \sqrt{3}, -1)$

SOLUTION

$x^2 + 2y^2 - 6x + 4y + 7 = 0$

Collect the like terms

$x^2 - 6x + 2y^2 + 4y = -7$

$x^2 - 6x + 2(y^2 + 2y) = -7$

We shall apply completing the square method. Follow the given step below

I) Find half of coefficient of x and y

II) Square them

III) add their value to the equation and subtract the same value again

$x^2 - 6x + 3^2 - 3^2 + 2(y^2 + 2y + 1^2 - 1^2) = -7$

open the bracket

$x^2 - 6x + 3^2 - 3^2 + 2y^2 + 4y + 2 - 2 = -7$

Take the negative part to R.H.S

$x^2 - 6x + 3^2 + 2y^2 + 4y + 2 = -7 + 3^2 + 2$

$x^2 - 6x + 9 + 2(y^2 + 2y + 1) = -7 + 9 + 2$

write it in expansion form

$(x-3)^2 + 2(y+1)^2 = 4$

divide through by 4

$\frac{(x-3)^2}{4} + \frac{\cancel{2}(y+1)^2}{\cancel{4}} = 1$

$\frac{(x-3)^2}{4} + \frac{(y+1)^2}{2} = 1$

$\frac{(x-\bar{x})^2}{a^2} + \frac{(y-\bar{y})^2}{b^2} = 1$

$a^2 = 4$ $b^2 = 2$ $(\bar{x}, \bar{y}) = (3, -1)$

$c^2 = a^2 - b^2 = 4 - 2 = 2$, $c^2 = 2$

$c = \pm\sqrt{2}$

The equation has its centre at the coordinate $(\bar{x}, \bar{y})$ not at the origin $(0, 0)$. Therefore

Foci $= (h \pm c, k)$ then

$(h, k) = (\bar{x}, \bar{y}) = (3, -1)$

Foci $= (3 \pm \sqrt{2}, -1)$

Answer is (C)

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TRUTH: The Lord will destroy the house of the proud (Proverb 15:25)

Given the ellipse $\frac{(x-1)^2}{21} + \frac{(y-3)^2}{4} = 1$

No 27: It has centre at (A) $(-1,3)$ (B) $(1,3)$ (C) $(1,-3)$ (D) $(-1,-3)$ (E) $(2,-3)$

SOLUTION

$$\frac{(x-1)^2}{21} + \frac{(y-3)^2}{4} = 1$$

$$\frac{(x-\bar{x})^2}{a^2} + \frac{(y-\bar{y})^2}{b^2} = 1 \quad (B)$$

$(\bar{x},\bar{y}) = (1,3)$ $a^2 = 21$, $b^2 = 4$

28) It has foci of (A) $(6,3)$ $(-4,3)$ (B) $(3,6)$ $(3,-4)$ (C) $(8,1)$ $(-2,1)$ (D) $(1,8)$ $(1,-2)$ (E) None

SOLUTION

foci $= (h \pm c, k)$, $(h,k) = (\bar{x},\bar{y}) = (1,3)$

$c^2 = a^2 - b^2 = 21 - 4 = 17$

$c^2 = 17$ $c = \sqrt{17}$ (E)

foci $= (h \pm c, k) = (1 \pm \sqrt{17}, 3)$ or

foci $= (1+\sqrt{17}, 3)$ $(1-\sqrt{17}, 3)$

19

2013 No 6: The equation $3x^2+2y^2+7x+14y+5=0$ represent (A) a parabola (B) a hyperbola (C) an ellipse (D) a circle (E) rectangular hyperbola

SOLUTION

CHARACTERISTICS OF ELLIPSE EQUATION

I) x and y have the same highest power i.e x^2 and y^2

II) Coefficient of x^2 and y^2 are not equal

III) Coefficient of x^2 and y^2 are positive value

IV) No xy terms

(C)

No 13 The equation of a conic with focus at $(0,0)$, directrix $y+1=0$ and eccentricity $1/2$ is (A) $4x^2+4y^2=1$ (B) $3x^2-4y^2+2y=1$ (C) $4x^2-3y^2+2y=1$ (D) $4x^2+3y^2=2y+1$ (E) $3x^2+4y^2=8(x+y-1)$

SOLUTION

Focus $= (0,0)$ at point (x,y)

$$|\text{Focus}| = \sqrt{(x-x_1)^2+(y-y_1)^2}$$

$$= \sqrt{(x-0)^2+(y-0)^2}$$

$$= \sqrt{x^2+y^2}$$

directrix $y+1=0$ at point (x,y)

Compare with $ax+by+c=0$

$a=0$, $b=1$, $c=1$

$$|\text{directrix}| = \frac{ax+by+c}{\sqrt{a^2+b^2}}$$

$$= \frac{0\times x + 1\times y + 1}{\sqrt{0^2+1^2}} = \frac{y+1}{1} = \underline{y+1}$$

$$e = \frac{|\text{Focus}|}{|\text{directrix}|}$$

$$\frac{1}{2} = \frac{\sqrt{x^2+y^2}}{y+1}$$

Square both side

$$\left(\frac{1}{2}\right)^2 = \frac{(\sqrt{x^2+y^2})^2}{(y+1)^2}$$

$$\frac{1}{4} = \frac{x^2+y^2}{(y+1)^2}$$

$$\frac{(y+1)^2}{4} = \frac{x^2+y^2}{1}$$

$$y^2+1+2y = 4(x^2+y^2)$$

$$4x^2+4y^2 = y^2+2y+1$$

$$4x^2+4y^2-y^2 = 2y+1$$

$$4x^2+3y^2 = 2y+1$$

(D)

21) The equation of a conic whose focus is at $(1,1)$, directrix $x=0$ and eccentricity $1/2$ is

(A) $\frac{x^2}{19}+\frac{y^2}{21}=1$ (B) $19x^2+4y^2-20xy+20x=1$
(C) $3x^2+4y^2-8(x+y-1)=0$
(D) $20x^2-20xy-x=1$ (E) $x^2+4y^2-xy=0$

SOLUTION

focus $=(1,1)$ at point (x,y)

$$|\text{Focus}| = \sqrt{(x-x_1)^2+(y-1)^2}$$
$$= \sqrt{(x-1)^2+(y-1)^2}$$
$$= \sqrt{x^2+1-2x+y^2+1-2y}$$
$$= \sqrt{x^2+y^2-2x-2y+2}$$

directrix $x=0$

$|\text{directrix}| = x$

$$e = \frac{|\text{Focus}|}{|\text{directrix}|}$$

20

$$\frac{1}{2} = \frac{\sqrt{x^2+y^2-2x-2y+2}}{x}$$

$$\frac{x}{2} = \sqrt{x^2+y^2-2x-2y+2}$$

Square both side

$$\left(\frac{x}{2}\right)^2 = x^2+y^2-2x-2y+2$$

$$\frac{x^2}{4} = x^2+y^2-2x-2y+2$$

$$x^2 = 4(x^2+y^2-2x-2y+2)$$ C

$$x^2 = 4x^2+4y^2-8x-8y+8$$

$$4x^2-x^2+4y^2+8(-x-y+1)=0$$

$$3x^2+4y^2-8(x+y-1)=0$$

10) The eccentricity of a conic with directrix $y=2x+3$, focus $(1,1)$ and which passes through the point $(3,2)$ is (A) $7/5$ (B) $5/2$ (C) 7 (D) 5 (E) $-2/5$

SOLUTION

directrix $y=2x+3$ at point $(3,2)$

$$y-2x-3=0$$

Compare with $ax+by+c=0$

$$a=-2,\ b=1,\ c=-3$$

$(x,y)=(3,2)$

$$|\text{directrix}| = \frac{ax+by+c}{\sqrt{a^2+b^2}}$$

$$|\text{directrix}| = \frac{-2\times3+1\times2+(-3)}{\sqrt{(-2)^2+1^2}}$$

$$= \frac{-6+2-3}{\sqrt{4+1}} = \frac{-7}{\sqrt{5}}$$

Focus $=(1,1)$ at point $(3,2)$

$$|\text{focus}| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$
$$= \sqrt{(3-1)^2+(2-1)^2} = \sqrt{2^2+1^2}$$
$$= \sqrt{4+1} = \sqrt{5}$$

$$e = \frac{|\text{Focus}|}{|\text{directrix}|} = \frac{\sqrt{5}}{-7/\sqrt{5}}$$

$$e = \frac{\sqrt{5}\times\sqrt{5}}{-7} = \frac{5}{-7}$$

$$|e| = \left|\frac{5}{-7}\right| = \frac{5}{7} < 1$$

22) The equation of directrices of the ellipse $\frac{(x-1)^2}{4}+\frac{(y+1)^2}{1}=1$ are

(A) $x=\pm\frac{4\sqrt{3}}{3}$ (B) $x=2\pm\frac{4\sqrt{3}}{3}$
(C) $y=1\pm\frac{4\sqrt{3}}{3}$ (D) $y=2\pm\frac{4\sqrt{3}}{3}$
(E) $x=1\pm\frac{4\sqrt{3}}{3}$

SOLUTION

$$\frac{(x-1)^2}{4}+\frac{(y+1)^2}{1} \qquad a>b$$

Type A Ellipse

equation of directrix

$$= X = x-\bar{x} = \pm a/e$$

$a^2=4$ $a=\sqrt{4}=2$ $b^2=1$ $b=1$

$$b^2 = a^2(1-e^2)$$
$$1 = 4(1-e^2)$$
$$1/4 = (1-e^2)$$
$$e^2 = 1-1/4 = 3/4$$
$$e = \sqrt{3/4} = \pm\frac{\sqrt{3}}{2}$$
$$a/e = \frac{2}{\sqrt{3}/2} = \frac{4}{\sqrt{3}}$$

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TRUTH: Folly is joy to him that is destitute of wisdom (Proverb 15:21)

$\frac{a}{e} = 4/\sqrt{3}$ rationalise the denominator

$a/e = \frac{4\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \frac{4\sqrt{3}}{\sqrt{3}}$ (C)

$x - \bar{x} = \pm a/e \quad x-1 = \pm \frac{4\sqrt{3}}{3}$

$x = 1 \pm \frac{4\sqrt{3}}{3}$

(28) The equation of an ellipse that has the length of its major axis as 6 and foci at (-2, 0) and (2, 0) is (A) $\frac{x^2}{9} - \frac{y^2}{5} = 1$ (B) $\frac{x^2}{5} + \frac{y^2}{9} = 1$ (C) $\frac{x^2}{9} + \frac{y^2}{5} = 1$ (D) $\frac{x^2}{5} - \frac{y^2}{9} = 1$ (E) $\frac{x^2}{4} + \frac{y^2}{5} = 1$

SOLUTION

Foci = $(\pm c, 0) = (\pm 2, 0)$

$c = 2$

length of major axis is $= 2a = 6$

$a = 6/2 = 3$

$c^2 = a^2 - b^2$ or $b^2 = a^2 - c^2$

$b^2 = 3^2 - 2^2 = 9 - 4 = 5$ (21)

$\frac{x^2}{a^2} + \frac{y}{b^2} = 1$

$\frac{x^2}{3^2} + \frac{y}{5} = 1$ (C)

$\frac{x^2}{9} + \frac{y}{5} = 1$

2012: No 17 The foci of the ellipse $4y^2 + 5x^2 = 20$ are (A) (0, 0) (B) $(1, \pm 2)$ (C) $(0, \pm 1)$ (D) $(\pm 1, 2)$ (E) $(\pm 1, 0)$

SOLUTION

$4y^2 + 5x^2 = 20$

divide through by 20

$\frac{4y^2}{20} + \frac{5x^2}{20} = \frac{20}{20}$

$\frac{y^2}{5} + \frac{x^2}{4} = 1$

$\frac{x^2}{4} + \frac{y^2}{5} = 1 \quad a > b \quad 5 > 4$

This is type B ellipse who foci $= (0, \pm c)$

$a^2 = 5, \; b^2 = 4$

$c^2 = a^2 - b^2 = 5 - 4 = 1$ (C)

$c^2 = 1, \; c = \sqrt{1} = 1$

Focus $= (0, \pm c) = (0, \pm 1)$

(18) The vertexes of the ellipse $4x^2 + 18y^2 = 72$ are (A) $(\pm 3, 0)$ \$ (0, 0) (B) $(\pm 3\sqrt{2}, 0)$ \$ $(0, \pm 2)$ (C) $(\pm\sqrt{3}, 0)$ \$ $(0, \sqrt{2})$ \$ (0, 2) (D) $(\pm 2, 0)$ \$ $(0, \sqrt{2})$ (E) (2, 0) \$ (0, 2)

SOLUTION

$4x^2 + 18y^2 = 72$

divide through by 72

$\frac{4x^2}{72} + \frac{18y^2}{72} = \frac{72}{72}$

$\frac{x^2}{18} + \frac{y^2}{4} = 1$

Vertices $= (\pm a, 0)$

Convertices $= (0, \pm b)$ (B)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a^2 = 18 \quad a = \sqrt{18} = \sqrt{9\times 2} = 3\sqrt{2}$

$b^2 = 4 \quad b = \sqrt{4} = 2$

Vertices $= (\pm 3\sqrt{2}, 0)$ Convertices $= (0, \pm 2)$

2009: No 14: The coordinate of the vertices of the curve described by the equation $2x^2 + y^2 - 4x - 14 = 0$ are (A) (4, 2) and (-4, 2) (B) (1, 4) and (1, -4) (C) (2, -4) and (2, 4) (D) (4, 1) and (-4, 1) (E) (4, 0) and (-4, 0)

SOLUTION

$2x^2 + y^2 - 4x - 14 = 0$

Collect the like terms -

$2x^2-4x+y^2-14=0$

$2(x^2-2x)+y^2=14$

find half of coefficient of x, square it add it and subtract it again from LHS

$2(x^2-2x+1^2-1^2)+y^2=14$

$2x^2-4x+1\cdot 2-2\cdot 1+y^2=14$

$2x^2-4x+2+y^2=14+2$

$2(x^2-2x+1)+y^2=16$

$2(x-1)^2+y^2=16$

divide through by 16

$\frac{\cancel{2}(x-1)^2}{\cancel{16}}+\frac{y^2}{16}=\frac{16}{16}$ This is type

$\frac{(x-1)^2}{8}+\frac{y^2}{16}=1$ B ellipse $a>b$, $16>8$

vertices $=(0,\pm a)$ which means from the ellipse equation

$x-1=0$, $y=\pm a$

$a^2=16$ $a=\sqrt{16}=4$

$x=1$, $y=\pm 4$ (B)

vertices $=(1,4)$ and $(1,-4)$

2008 · The foci and directrix of the ellipse $4x^2+9y^2=36$ are respectively

(A) $(\pm 5,0)$, $x=3/5$ (B) $(\pm\sqrt{5},0)$ $x=\pm\frac{9}{\sqrt{5}}$

(C) $(0,\pm 8)$ $x=\pm\frac{9}{5}$ (D) $(\pm 5,0)$ $x=\pm\frac{9}{5}$

(E) $(0,\pm 5)$, $x=-3$

SOLUTION

$4x^2+9y^2=36$

divide through by 36

$\frac{\cancel{4}x^2}{\cancel{36}}+\frac{9y^2}{36}=\frac{36}{36}$

$\frac{x^2}{9}+\frac{y^2}{4}=1$

$c^2=a^2-b^2=9-4=5$

$c^2=5$, $c=\sqrt{5}$ = foci = $(\pm c, 0)$

foci $=(\pm\sqrt{5},0)$

Equation of directrix is $x=\pm a/e$

$a^2=9$ $a=\sqrt{9}=3$ (B)

$ae=c$ or $3\times e=\sqrt{5}$ then

$e=\sqrt{5}/3$ $x=\pm a/e=\pm\frac{3}{\sqrt{5}/3}=\pm\frac{9}{\sqrt{5}}$

EQUATION OF TANGENT TO ELLIPSE

$y=mx+c$ ---- eqn (i)

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ---- eqn (ii)

If eqn (i) is tangent to eqn (ii)

The equation of tangent to ellipse is

$\boxed{y=mx\pm\sqrt{a^2m^2+b^2}}$ where

$\boxed{c=\sqrt{a^2m^2+b^2}}$

Given a coordinate (x_1,y_1) of a line which is tangent to ellipse, The equation of tangent to ellipse is

$\boxed{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}$ or $\boxed{\frac{xx_1}{b^2}+\frac{yy_1}{a^2}=1}$

TYPE A TYPE B

EQUATION OF NORMAL TO THE ELLIPSE

If eqn (i) above is normal or perpendicular to eqn (ii) above equation of normal to the ellipse is

$\boxed{y=\frac{-1}{m}x\pm\sqrt{\frac{a^2}{m^2}+b^2}}$

$\boxed{c=\sqrt{\frac{a^2}{m^2}+b^2}}$

Given a coordinate of line (x_1,y_1) equation of normal to the ellipse is

$\frac{y-y_1}{y_1}=\frac{a^2}{b^2}\frac{(x-x_1)}{x_1}$ TYPE A Ellipse

$\frac{y-y_1}{y_1}=\frac{b^2}{a^2}\frac{(x-x_1)}{x_1}$ TYPE B ELLIPSE

Locus of points of intersection of two perpendicular tangent to the ellipse at centre $(0,0)$ is called director. (which is a circle) with the a locus

$x^2+y^2=a^2-b^2$

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TRUTH: Righteousness exalt a nation but sin is a reproach to any people (Proverb 14:34)

SOLVED PAST QUESTIONS AND ANSWERS

2013 No 29: The equation of a tangent to the ellipse $\frac{(x+3)^2}{3} + \frac{(y-1)^2}{2} = 1$ at the point $(-3, 4+\sqrt{2})$ is (A) $x = 4+\sqrt{2}$ (B) $y = 4+\sqrt{2}$ (C) $x = -3$ (D) $y = 3$ (E) $x = 4-\sqrt{2}$

SOLUTION

$$\frac{(x+3)^2}{3} + \frac{(y-1)^2}{2} = 1$$

$$2(x+3)^2 + 3(y-1)^2 = 6$$

$$2(x^2+6x+9) + 3(y^2-2y+1) = 6$$

$$2x^2+12x+18+3y^2-6y+3 = 6$$

Applying diffrentiation method, we differentiate both side with respect to x

$$4x+12+6y\frac{dy}{dx} - 6\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(6y-6) = -4x-12$$

$$m = \frac{dy}{dx} = \frac{-4x-12}{6y-6} = \frac{-2x-6}{3y-3}$$

at $(-3, 4+\sqrt{2})$

Equation of tangent is given as

$$y-y_1 = m(x-x_1) \text{ at } (-3, 4+\sqrt{2})$$

$$y-4-\sqrt{2} = \frac{-2(-3)-6}{3(4+\sqrt{2})-3}(x--3)$$

$$y-4-\sqrt{2} = \frac{+6-6}{12+3\sqrt{2}-3}(x+3)$$

$$y-4-\sqrt{2} = \frac{0}{12+3\sqrt{2}-3}(x+3)$$

$$y-4-\sqrt{2} = 0 \quad \text{B}$$

$$y = 4+\sqrt{2}$$

2012: The equation of the normal to the ellipse $4x^2+9y^2 = 36$ at the point $(0,-2)$ (A) $y=0$ (B) $x=5$ (C) $x=-2$ (D) $y=-2$ (E) $x=3$

SOLUTION

Method I $\quad 4x^2+9y^2 = 36$

differentiate both side with respect to x

$$8x+18y\frac{dy}{dx} = 0 \quad \text{at } (0,-2)$$

$$18y\frac{dy}{dx} = -8x$$

$$m = \frac{dy}{dx} = \frac{-8x}{18y} = \frac{-8(0)}{18(-2)} = 0$$

$$y-y_1 = m(x-x_1) \quad \text{C}$$

$$y--2 = 0(x-0)$$

$$y+2 = 0$$

$$y = 0-2 \quad \underline{y=-2}$$

Method II $\quad 4x^2+9y^2 = 36$

divide through by 36

$$\frac{4x^2}{36} + \frac{9y^2}{36} = \frac{36}{36}$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Equation of tangent at point (x,y)

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{at } (0,-2)$$

$$\frac{0\times x}{9} + \frac{y\times -2}{4} = 1 \quad \text{C}$$

$$0 + \frac{(-2y)}{4} = 1$$

$$-2y = 4$$

$$y = \frac{4}{-2} \quad y = -2$$

(22) The equation of the normal to the ellipse $x^2+4y^2 = 4$ at the point $(0,1)$ (A) $x=-4$, (B) $x=\pm 1$ (C) $x=0$ (D) $y=0$ (E) $y=\pm 1$

SOLUTION

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Method I $x^2+4y^2=4$

differentiate both side with respect to x

$$2x+8y\frac{dy}{dx}=0$$

$$8y\frac{dy}{dx}=2x$$

$$m=\frac{dy}{dx}=\frac{2x}{8y} \quad at\ (0,1)$$

$$m=\frac{2(0)}{8(1)}=0$$

$$y-y_1=\frac{-1}{m}(x-x_1) \quad \text{For normal}$$

$$y-1=\frac{-1}{0}(x-0)$$

$$0\times(y-1)=-1(x-0)$$

$$0=-x+0 \quad or\ x=0$$

Method II

$$x^2+4y^2=4$$

divide through by 4

$$\frac{x^2}{4}+\frac{4y^2}{4}=\frac{4}{4}$$

$$\frac{x^2}{4}+\frac{y^2}{1}=1$$

24

Equation of normal at point (x,y) is

$$\frac{y-y_1}{y_1}=\frac{a^2}{b^2}\frac{(x-x_1)}{0} \quad at\ (0,1)$$

$$\frac{y-1}{1}=\frac{4}{1}\frac{(x-0)}{0}$$

C

$$\frac{y-1}{1}=\frac{4x}{0}$$

$$4x=y-1\times 0$$

$$4x=0$$

$$x=0$$

2009: The value of k for which the line $y=\frac{1}{2}x+k$ is tangent to the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ are

(A) ± 4 (B) ± 2 (C) $\pm 2\sqrt{2}$ (D) ± 3 (E) ± 8

SOLUTION

$$y=\frac{1}{2}x+k$$

$$y=mx+c \qquad m=\tfrac{1}{2},\ c=k$$

$$\frac{x^2}{4}+\frac{y^2}{3}=1 \qquad a^2=4,\ b^2=3$$

For tangent $c=\sqrt{a^2m^2+b^2}$

$$k=\sqrt{4\left(\tfrac{1}{2}\right)^2+3}=\sqrt{4\times\tfrac{1}{4}+3}$$

$$k=\sqrt{1+3}=\pm\sqrt{4}=\pm 2$$

2008: No 10: The equation of the normal line to the ellipse $3x^2+4y^2=12$ parallel to the line $y=2x+k$ is (A) $y=2x\pm\sqrt{2}$ (B) $y=2x\pm 2$ (C) $y=\frac{1}{2}x-3$ (D) $y=-2x-3$ (E) None

SOLUTION

$$3x^2+4y^2=12$$

divide through by 12

$$\frac{3x^2}{12}+\frac{4y^2}{12}=\frac{12}{12}$$

$$\frac{x^2}{4}+\frac{y^2}{3}=1 \qquad a^2=4\quad b^2=3$$

$$y=2x+k \qquad m=2$$

$$y=mx+c \qquad c=k$$

Equation of normal is $y=\frac{-1}{m}x\pm\sqrt{\frac{a^2}{m^2}+b^2}$

$$y=\frac{-1}{2}x\pm\sqrt{\frac{4}{2^2}+3}$$

$$y=\frac{-1}{2}x\pm\sqrt{1+3}=\frac{-1}{2}x\pm\sqrt{4}$$

$$y=-\frac{1}{2}x\pm 2 \quad (E)$$

EXERCISE

Find the equation of (a) tangent (b) Normal to the ellipse $\frac{x^2}{16}+\frac{y^2}{1}=1$ which make an angle of $60°$ with the x-axis.

Hint $m=\tan\theta=\tan 60$

Find the equation of tangent and normal at the point where $y=2$ on the ellipse $x^2+4y^2=5$

HINT

Substitute for $y=2$ in the ellipse equation to get x so as to have point (x,y). Proceed to any method you want either differential or algebraic

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TRUTH: The eyes of the Lord are in every place beholding the evil and the good (Proverb 15:3)

PARAMETRIC EQUATION OF ELLIPSE

Let $x = a\cos\theta$, $Y = b\sin\theta$ to find the equation of tangent and normal to the ellipse in parametric form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{xX}{a^2} + \frac{yY}{b^2} = 1$$

$$\frac{x(\cancel{a}\cos\theta)}{\cancel{a^2}} + \frac{y(\cancel{b}\sin\theta)}{b^2} = 1$$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

Multiply through by ab

Parametric equation for tangent $\Rightarrow$ $\boxed{xb\cos\theta + ya\sin\theta = ab}$

Parametric equation of Normal $\Rightarrow$ $\boxed{\frac{ax\sin\theta - by\cos\theta}{[a^2-b^2]} = \sin\theta\cos\theta}$

2008: No 11: The cartesian equation of the curve described by the pair of equation $x = 2\sin\theta - 1$, $y = 1 + 3\cos\theta$ is (A) $\frac{(x+1)^2}{4} + \frac{(y-1)^2}{9} = 1$ (B) $\frac{(x-1)^2}{4} + \frac{(y-1)^2}{9} = 1$ (C) $\frac{(x+1)^2}{9} + \frac{(y-1)^2}{4} = 1$ (D) $\frac{(x-1)^2}{2} + \frac{(y-1)^2}{9} = 1$ (E) $\frac{(x+1)^2}{4} - \frac{(y-1)^2}{3} = 1$

SOLUTION

$x = 2\sin\theta - 1$ and $y = 1 + 3\cos\theta$

$2\sin\theta = x+1$ and $3\cos\theta = y-1$

$\sin\theta = \frac{x+1}{2}$ and $\cos\theta = \frac{y-1}{3}$

Recall $\sin^2\theta + \cos^2\theta = 1$

$$\left(\frac{x+1}{2}\right)^2 + \left(\frac{y-1}{3}\right)^2 = 1$$

$$\frac{(x+1)^2}{4} + \frac{(y-1)^2}{9} = 1 \quad (A)$$

2009 NO 12: The cartesian equation of the curve described by the pair of equations $x = 5\cos\phi$, $y = 3\sin\phi - 2$ is (A) $\frac{x^2}{25} + \frac{(y+2)^2}{9} = 1$ (B) $\frac{x^2}{25} - \frac{(y+2)^2}{9}$ (C) $\frac{(x-2)^2}{9} + \frac{y^2}{25} = 1$ (D) $\frac{x^2}{5} - \frac{(y-2)^2}{3} = 1$ (E) None of the above

SOLUTION

$x = 5\cos\phi$ and $y = 3\sin\phi - 2$

$\cos\phi = \frac{x}{5}$ and $3\sin\phi = y+2$

$\cos\phi = \frac{x}{5}$ and $\sin\phi = \frac{y+2}{3}$

$$\cos^2\phi + \sin^2\phi = 1$$

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y+2}{3}\right)^2 = 1 \quad (A)$$

$$\frac{x^2}{25} + \frac{(y+2)^2}{9} = 1$$

LOCUS OF MIDDLE POINT OF ELLIPSE

Locus of middle point of a system of parallel chord with slope m of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given as

$$y = -\frac{b^2}{a^2 m}x \quad \text{or} \quad x = -\frac{a^2 m}{b^2}y$$

EXERCISE

Find the equation of the tangent to the ellipse $x^2 + 2y^2 = 19$ which are parallel to the line $x + 6y = 5$

HINT

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

$x + 6y = 5$ $\quad y = -\frac{1}{6}x + \frac{5}{6}$ $\quad m = -1/6$

Find the equation of the tangent to the ellipse $x^2 + 4y^2 = 4$ which are perpendicular to the line $2x - 3y = 1$

HINT

$$y = -\frac{1}{m}x \pm \sqrt{a^2m^2 + b^2}$$

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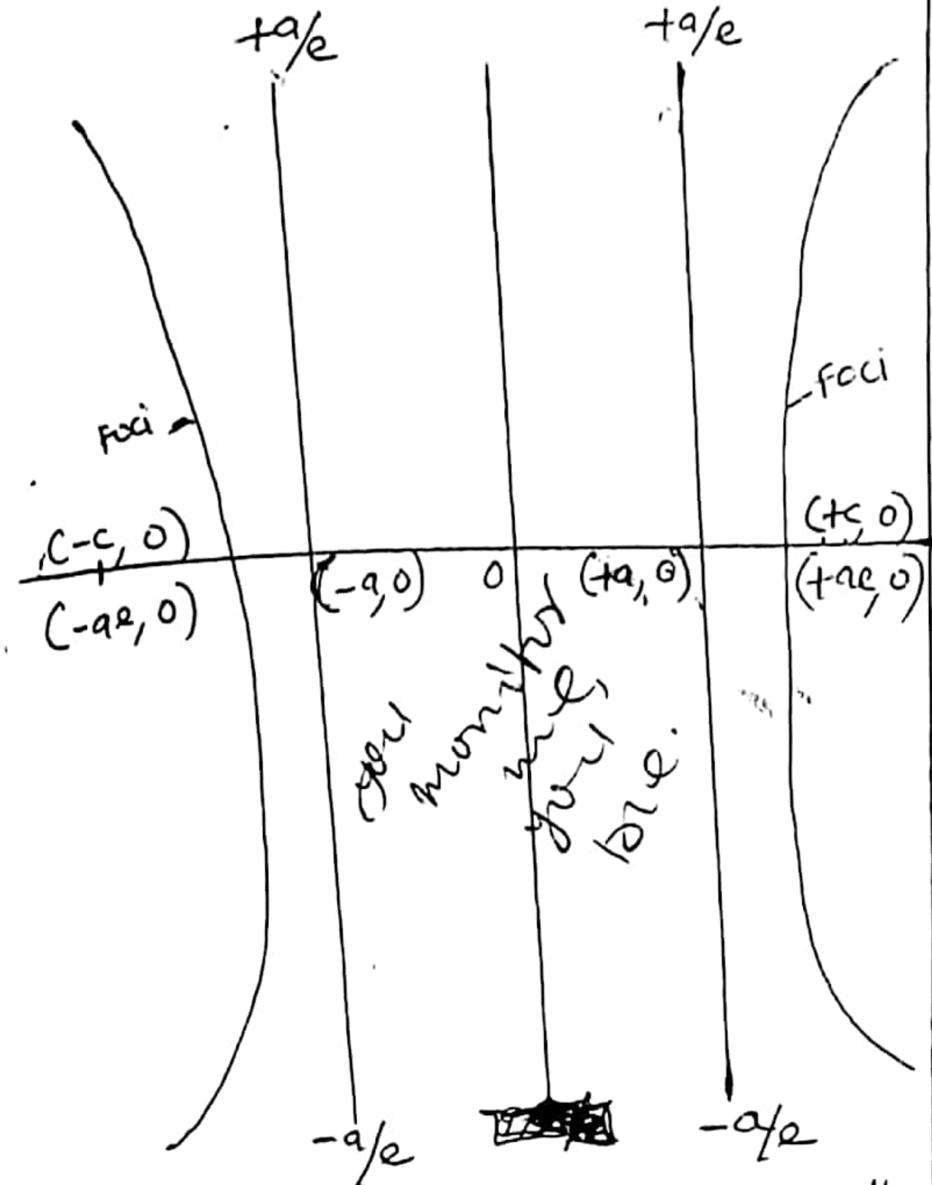
TRUTH: Better is little with the fear of the Lord than great treasure and trouble there with (Proverbs 15:16)

HYPERBOLA

TYPE A



The hyperbola has its equation with respect to origin (0, 0) given as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad b > a$$

If there are two given coordinates point that lies on the axis of the hyperbola $A(x_1, y_1)$ and (x_2, y_2). Then

$$(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

27

$$2ae = \sqrt{(x_1 - x_2)^2 + (y - y_1)^2}$$

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1$$

Foci $= (\pm c, 0) = (\pm ae, 0)$

$c^2 = a^2 + b^2$

Vertices $= (\pm a, 0)$

Convertices $= (0, \pm b)$

Equation of directrix is $x = \pm a/e$

Equation of asymptotes is given as

$$y = \pm \frac{b}{a}x \quad \text{or} \quad x = \pm \frac{a}{b}y$$

Two tangent are perpendicular to each other on the hyperbola curve when

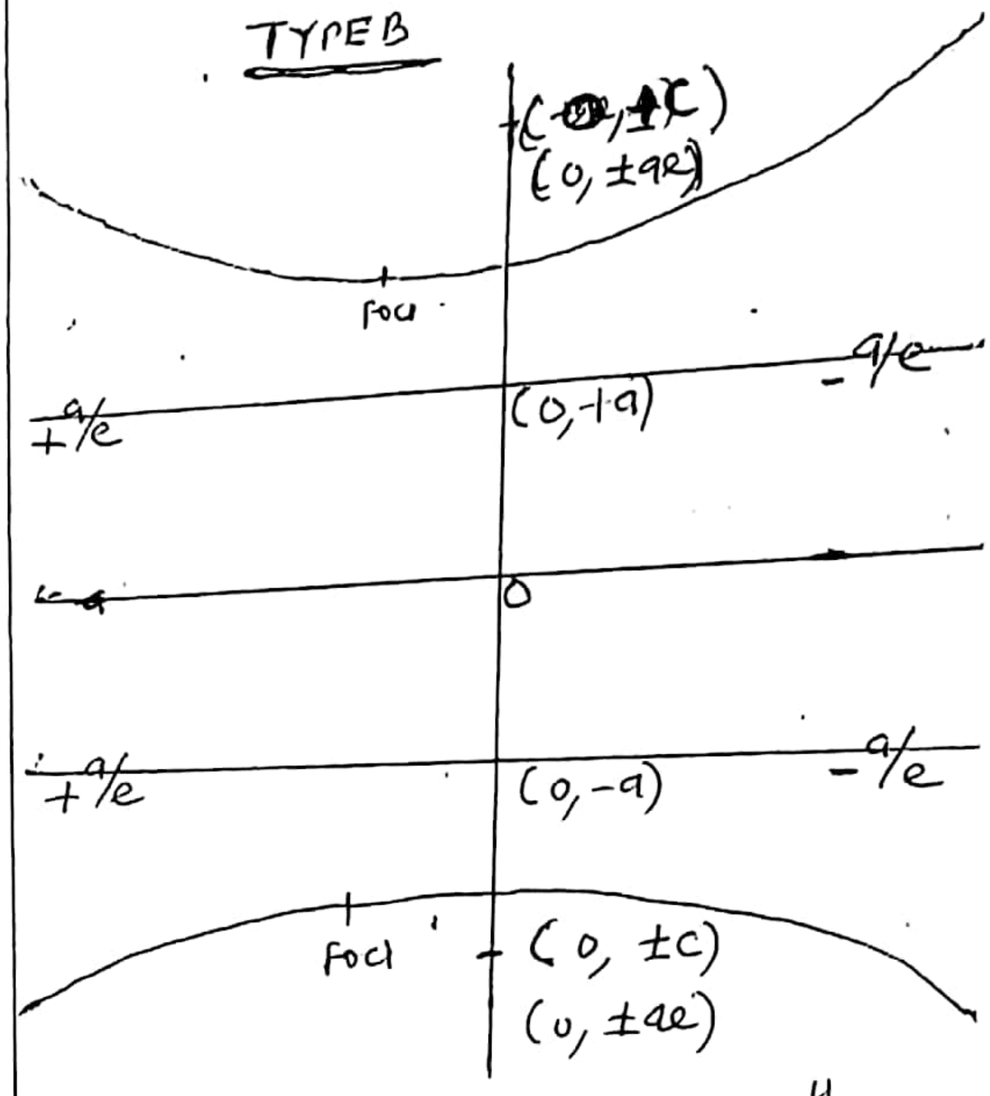
$$y = \frac{b^2}{a^2}x$$

equation of minor axis, $x = 0$

Equation of major axis $y = 0$

Coordinate of Lactum rectum is $= (\pm c, b^2/a)$ which are given as $(+c, b^2/a)$ and $(-c, b^2/a)$

TYPE B



The hyperbola has its equation with respect to origin given as

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1 \quad b > a$$

at coordinate $(x_0\ y_0)$ we have

$$\frac{(x - x_0)^2}{b^2} - \frac{(y - y_0)^2}{a^2} = 1$$

Foci $= (0, \pm c) = (0, \pm ae)$

Vertices $= (0, \pm a)$

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Co-vertices $= (\pm b, 0)$

Coordinate of lactum rectum is given as $(c, \pm b^2/a)$ which are $(c, +b^2/a)$ and $(c, -b^2/a)$

Equation of major axis $x = 0$

Equation of minor axis $y = 0$

Equation of directrix $y = \pm a/e$

Equation of asymptotes $x = \pm \frac{b}{a} y$ or

$$y = \pm \frac{a}{b} x$$

Two tangent are perpendicular to each other on the hyperbola curve when

$$y = \frac{a^2}{b^2} x$$

GENERAL PROPERTIES OF HYPERBOLA

Given two coordinate points $A(x_1, y_1)$ and $B(x_2, y_2)$ on hyperbola curve

$$(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$c = 2ae = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Length of lactum rectum $= 2b^2/a$

Length of transverse axis $= 2a$

Length of conjugate axis $= 2b$

$$b^2 = a^2(e^2 - 1) \quad e > 1$$

$$c = \sqrt{a^2 + b^2}$$

Coordinate of lactum rectum are $(ae, b^2/a)$ and $(-ae, b^2/a)$ and $(ae, -b^2/a)$ and $(ae, b^2/a)$

Equation of tangent to ellipse is given

$$\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1 \quad \text{type A} \qquad \frac{x x_1}{b^2} - \frac{y y_1}{a^2} = 1 \quad \text{type B}$$

Equation of normal to the ellipse

$$\frac{y - y_1}{y_1} = \frac{-a^2}{b^2}\frac{(x - x_1)}{x_1} \quad \text{type A}$$

$$\frac{y - y_1}{y_1} = \frac{-b^2}{a^2}\frac{(x - x_1)}{x_1} \quad \text{type B}$$

EXAMPLE

Find the coordinate of the foci of hyperbola $\frac{(x-1)^2}{9} - \frac{(y-3)^2}{16} = 1$

SOLUTION

$$\frac{(x-1)^2}{9} - \frac{(y-3)^2}{16} = 1 \quad b > a \quad \text{type A}$$

$a^2 = 9 \qquad b^2 = 16$

$a = \sqrt{9} \qquad b = \sqrt{16}$

$a = 3 \qquad b = 4$

$c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

$c = ae = 5$ or $3 \times e = 5$, $e = 5/3$

Foci $= (\pm c, 0) = (\pm 5, 0) = (\bar{X}, \bar{Y})$

where $\bar{X} = x - 1 = \pm 5$ and $\bar{Y} = y - 3 = 0$

$x = 1 \pm 5$ and $y = 0 + 3$

$x = 1 + 5$ or $1 - 5$ and $y = 3$

$x = 6$ or -4 and $y = 3$

Coordinate of foci $= (6, 3)$ and $(-4, 3)$

EXAMPLE

28

Find the equation of the asymptotes of the curve $\frac{x^2}{9} - \frac{y^2}{4} = 1$

SOLUTION

$$\frac{x^2}{9} - \frac{y^2}{4} = 1 \quad b > a \quad \text{(TYPE B)}$$

$b^2 = 9 \quad b = \sqrt{9} = 3$

$a^2 = 4 \quad a = \sqrt{4} = 2$

For type B hyperbola, the equation of asymptotes is

$$y = \pm \frac{a}{b} x$$

$$y = \pm \frac{2}{3} x$$

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TRUTH: Evil pursueth sinners but to the righteous good shall be repaid (Proverbs 13: 21)

TANGENT AND NORMAL TO HYPERBOLA

Given an hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $b > 1$

and a straight line $y = mx + c$

Equation of tangent to hyperbola is

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

where $c = \sqrt{a^2m^2 - b^2}$

note: c is called intercept on y axis

Equation of normal to hyperbola is

$$y = -\frac{1}{m}x \pm \sqrt{\frac{a^2}{m^2} - b^2}$$

where $c = \sqrt{\frac{a^2}{m^2} - b^2}$

PARAMETRIC EQUATION OF HYPERBOLA

Given $X = a\sec\theta$ and $Y = b\tan\theta$

at point $(x, y) = (a\sec\theta, b\tan\theta)$

Equation of tangent to hyperbola

is $\boxed{(b\sec\theta)x - (a\tan\theta)y = ab}$

Equation of normal to hyperbola

is $\boxed{\frac{b}{\tan\theta}y + \frac{a}{\sec\theta}x = a^2 + b^2}$

or

$(b\sec\theta)x + (a\tan\theta)y = (a^2+b^2)\sec\theta\tan\theta$

SOLVED PAST QUESTIONS AND ANSWERS

2015 : NO 7. The value of Q such that line $y = -3x + Q$ is a tangent to the conics $\frac{x^2}{49} - \frac{y^2}{49} = 1$ is (29)

(A) $\pm 14\sqrt{2}$ (B) ± 14 (C) $\pm 7\sqrt{2}$ (D) $\pm 7\sqrt{10}$

(E) NONE

SOLUTION

$$\frac{x^2}{49} - \frac{y^2}{49} = 1$$

$a^2 = 49$ $b^2 = 49$

$y = -3x + Q$ $m = -3$

$y = mx + c$ $c = Q$

$c = \sqrt{a^2m^2 - b^2}$

$c = \sqrt{49(-3)^2 - 49}$ (A)

$c = \sqrt{49 \times 9 - 49}$

$c = \sqrt{49(9-1)} = \sqrt{49 \times 8}$

$c = 7 \times \sqrt{8} = 7 \times \sqrt{4 \times 2}$

$c = 7 \times 2\sqrt{2} = \pm 14\sqrt{2}$

$Q = c = \pm 14\sqrt{2}$

2012 : NO 26 The tangent of the hyperbola at the point $x^2 - 2y^2 = 6$ at the point $(2\sqrt{3}, \sqrt{3})$ is

(A) $y + 2x - 6 = 0$ (B) $y + x = 5$

(C) $y - x + \sqrt{3} = 0$ (D) $y - 2x + 4 = 0$

(E) $y - x = 9$

SOLUTION

$$x^2 - 2y^2 = 6$$

$$\frac{x^2}{6} - \frac{2y^2}{6} = 1$$

$$\frac{x^2}{6} - \frac{y^2}{3} = 1 \qquad (x_1, y_1) = (2\sqrt{3}, \sqrt{3})$$

Equation of tangent at (x_1, y_1) $= \frac{xx_1}{b^2} - \frac{yy_1}{a^2} = 1$

$$\frac{(2\sqrt{3})x}{6} - \frac{\sqrt{3}y}{3} = 1$$

multiply through by 6

$$2\sqrt{3}x - 2\sqrt{3}y = 6$$

divide through by 2

$$\sqrt{3}x - \sqrt{3}y = 3$$

$$\sqrt{3}(x - y) = 3$$

$x - y = \frac{3}{\sqrt{3}}$ rationalize the denominator

$$x - y = \frac{3 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$

$$x - y = \frac{3\sqrt{3}}{3}$$ C

$x - y = \sqrt{3}$ or $y - x + \sqrt{3} = 0$

EXAMPLE

Given a curve represented by the equation $x = 3\sec\theta - 1$ and $y = 4\tan\theta + 2$. Find

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a) eccentricity of the curve
(b) Intercept on x-axis
(c) ~~Equation~~ of minor axis
d) Coordinate of the foci
e) Coordinate of the ~~lactum~~ rectum
f) Length of major axis and minor axis
g) equation of directrix
h) Equation of asymptotes to the curve

SOLUTION

$x = 3\sec\theta - 1 \qquad y = 4\tan\theta + 2$

$x+1 = 3\sec\theta \qquad y-2 = 4\tan\theta$

$\sec\theta = \frac{x+1}{3} \qquad \tan\theta = \frac{y-2}{4}$

$1+\tan^2\theta = \sec^2\theta$

or $\sec^2\theta - \tan^2\theta = 1$

$\left(\frac{x+1}{3}\right)^2 - \left(\frac{y-2}{4}\right)^2 = 1$ type A hyperbola

$\frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$

(a) $b^2 = a^2(e^2-1)$ --- xxx $\quad a^2=9$, $b^2=16$

$e^2-1 = \frac{b^2}{a^2}$ then $e^2 = \frac{b^2}{a^2}+1$

$e^2 = \frac{16}{9}+1 \quad e^2 = \frac{25}{9} \quad e = \sqrt{\frac{25}{9}} = 5/3$

b) Intercept on x-axis is where the curve cut x-axis. Its known as vertices $= (\pm a, 0) = (\overline{X}, \overline{Y})$

$\overline{X} = x - \overline{x} = \pm a \quad \overline{Y} = y - \overline{y} = 0$

$x+1 = \pm 3 \quad y-2 = 0$

$x = -1\pm 3$ and $y = 0+2$

$x = -1+3$ or $-1-3$ and $y=2$

$x = 2$ or -4 and $y=2$

Coordinate of vertices $= (2,2)$ and $(-4,2)$

(c) ~~Equation~~ of minor axis

$\overline{X} = 0 \quad x-\overline{x} = 0 \quad x+1 = 0 \quad \underline{x=-1}$

Equation of major axis

$\overline{Y} = 0 \quad y-\overline{y} = 0, \quad y-2=0, \quad \underline{y=2}$

d) foci $= (\pm c, 0)$

$c = \sqrt{a^2+b^2} = \sqrt{9+16} = \sqrt{25}$

30

Foci $= (\pm 5, 0) = (\overline{X}, \overline{Y})$

$\overline{X} = x - \overline{x} = \pm 5 \quad \overline{Y} = y - \overline{y} = 0$

$x+1 = \pm 5$ and $y-2=0$

$x = -1\pm 5$ and $y = 0+2$

$x = -1+5$ or $-1-5$ and $y=2$

$x = 4$ or -6 and $y=2$

Coordinate of foci $= (4,2)$ and $(-6,2)$

(f) Length of major axis (transverse axis) $= 2a$

$= 2\times 3 = 6$

Length of minor axis (Conjugate axis) $= 2b$

$2\times 4 = 8$

(g) equation of directrix $\overline{X} = \pm a/e$

$\overline{X} = x - \overline{x} = \pm a/e, \quad x+1 = \pm \frac{3}{5/3}$

$x+1 = \pm 9/5$ then $x = -1\pm 9/5$

$x = -1+9/5$ or $x = -1-9/5$

$x = \frac{-5+9}{5}$ or $x = \frac{-5-9}{5}$

$x = 4/5$ or $x = -14/5$

(h) Equation of asymptotes is given as

$y = \pm\frac{b}{a}x$ then $y = \pm\frac{4}{3}x$

(e) Coordinate of lactum rectum $= (\pm c, \frac{b^2}{a})$

$= (\pm 5, 16/3) = (\overline{X}, \overline{Y})$

$\overline{X} = x - \overline{x} = \pm 5$ and $\overline{Y} = y - \overline{y} = 16/3$

$x+1 = \pm 5$ and $y-2 = 16/3$

$x = -1\pm 5$ and $y = 2+16/3 = 22/3$

$x = 4$ or -6 and $y = 22/3$

Coordinate of lactum rectum is $(4, 22/3)$ and $(-6, 22/3)$

Length of lactum rectum $= \frac{2b^2}{a} = \frac{2\times 16}{3} = \frac{32}{3}$

Assignment

Read on rectangular hyperbola

(i) its eccentricity

(ii) foci, vertices, directrix and so on

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TRUTH: In all labor there is profit but the talk of the lips tendeth only to penury (Proverb 14:23)

SOLVED PAST QUESTIONS AND ANSWERS

2015 No 4: Equation of asymptotes of the hyperbola $3x^2-4y^2+3x+16y-18=0$ are (A) $2y=\pm\sqrt{3}x$ (B) $\sqrt{2}y=\pm 3x$ (C) $4y=\pm 3x$ (D) $2x=\pm\sqrt{3}y$ (E) None

SOLUTION

$$3x^2-4y^2+3x+16y-18=0$$

There are two methods

a) Method I, re-write the equation

as $\frac{(x-x_0)^2}{b^2}-\frac{(y-y_0)^2}{a^2}=1$ $b>a$

$\boxed{\text{Type B hyperbola}}$

Equation of asymptotes is $y=\pm\frac{a}{b}x$

b) Method II: Equate the terms with highest power to zero

$$3x^2-4y^2=0$$

$$3x^2=4y^2 \text{ or } y^2=\frac{3}{4}x^2$$

Find the square root of both side

$$\sqrt{y^2}=\pm\sqrt{\frac{3}{4}x^2}$$

$$y=\pm\frac{\sqrt{3}}{2}x$$

$$2y=\pm\sqrt{3}x$$

(A)

No 9: The equation of the asymptotes of the conic $36x^2-64y^2=2304$ is

(A) $y^2=\pm\frac{6}{8}x$ (B) $y=\pm\frac{6}{8}x^2$ (C) $y=\pm\frac{3}{4}x$ (D) $x=\pm\frac{3}{4}y$ (E) None

SOLUTION

Method I $36x^2-64y^2=2304$

divide through by 2304

$$\frac{36x^2}{2304}-\frac{64y^2}{2304}=1$$

$$\frac{x^2}{64}-\frac{y^2}{36}=1 \quad b>a$$

$\boxed{\text{Type B hyperbola}}$

Equation of asymptotes $y=\pm\frac{a}{b}x$

$a^2=36$ $a=\sqrt{36}=6$

$b^2=64$ $b=\sqrt{64}=8$ (C)

$$y=\pm\frac{a}{b}x, \quad y=\pm\frac{6}{8}x$$

$$y=\pm\frac{3}{4}x$$

Method II

Equate the terms with highest power to zero $36x^2-64y^2=0$

$$64y^2=36x^2$$

$$y^2=\frac{36}{64}x^2$$

Find the square root of both side

$$\sqrt{y^2}=\sqrt{\frac{36}{64}x^2}$$

$$y=\pm\frac{6}{8}x \text{ or } y=\pm\frac{3}{4}x$$

(C)

22) A conic whose eccentricity equal to $\sqrt{2}$ is (A) parabola (B) rectangular hyperbola (C) ellipse (D) hyperbola (E) None

SOLUTION

$e=\sqrt{2}$ $(e>1)$ Answer is (D)

2014: No 5: The equation of hyperbola with vertices $(\pm 3,0)$, asymptotes $y=\pm 2x$ is (A) $\frac{x^2}{3}-\frac{y^2}{9}=1$ (B) $\frac{x^2}{6}-\frac{y^2}{3}=1$ (C) $\frac{x^2}{3}-\frac{y^2}{6}=1$ (D) $\frac{x^2}{36}-\frac{y^2}{9}=1$ (E) $\frac{x^2}{9}-\frac{y^2}{36}=1$

SOLUTION

The vertices $=(\pm 3,0)=(\pm a,0)$ shows that its a type A hyperbola then $a=3$ while equation of asymptotes is $y=\pm\frac{b}{a}x$ which is given above as $y=\pm 2x$

Therefore $\frac{b}{a}=2$ when $a=3$

$$b=2\times a=2\times 3=6$$

$$b=6 \quad a=3$$

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For type A hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad b > a$$

$$\frac{x^2}{3^2} - \frac{y^2}{6^2} = 1 \quad E$$

$$\frac{x^2}{9} - \frac{y^2}{36} = 1$$

2012: NO 25: The foci at the hyperbola $16x^2 - 3y^2 = 48$ is? (A) $(2\sqrt{3}, \sqrt{2})$ (B) $(\pm\sqrt{5}, \sqrt{2})$ (C) $(0, 7)$ (D) $(\pm 3\sqrt{9}, 0)$ (E) $(\pm\sqrt{19}, 0)$

SOLUTION

$16x^2 - 3y^2 = 48$

divide through by 48

$$\frac{16}{48}x^2 - \frac{3}{48}y^2 = 1$$

Type A hyperbola

$$\frac{x^2}{3} - \frac{y^2}{16} = 1 \quad b > a$$

$a^2 = 3 \quad b^2 = 16$

$$c = \sqrt{a^2 + b^2} = \sqrt{3 + 16}$$

$c = \sqrt{19}$ E

Foci $= (\pm c, 0) = (\pm\sqrt{19}, 0)$

2009 NO 13: The equation of the hyperbola with vertices $(0, \pm 5)$ and foci $(0, \pm\sqrt{26})$ is (A) $\frac{y^2}{25} - \frac{x^2}{26} = 1$ (B) $\frac{y^2}{25} - \frac{x^2}{144} = 1$ (C) $\frac{x^2}{25} - \frac{y^2}{1} = 1$ (D) $\frac{y^2}{25} - \frac{x^2}{1} = 1$ (E) $\frac{x^2}{26} - \frac{y^2}{25} = 1$

SOLUTION 32

vertices $= (0, \pm 5) = (0, \pm a)$ and

foci $= (0, \pm\sqrt{26}) = (0, \pm c)$

show that this is type B hyperbola

$a = 5 \quad c = \sqrt{26}$

$c^2 = a^2 + b^2$ or $b^2 = c^2 - a^2$

$b^2 = (\sqrt{26})^2 - 5^2 = 26 - 25 = 1$

$b^2 = 1, \quad a^2 = 25$

$$\frac{x^2}{25} - \frac{y^2}{1} = 1 \quad C$$

For every hyperbola y^2 is negative

(21) If $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is the equation of a hyperbola then the eccentricity is (A) $5/3$ (B) $5/4$ (C) $4/3$ (D) $4/5$ (E) $-5/3$

SOLUTION

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \quad b > a$$

$a^2 = 9 \quad b^2 = 16$ A

$b^2 = a^2(e^2 - 1)$

$e^2 - 1 = \frac{b^2}{a^2} = \frac{16}{9} \quad e^2 = \frac{16}{9} + 1 = \frac{25}{9}$

$e = \sqrt{25/9} = 5/3 > 1$

2010: NO 23: The equation of the hyperbola centered at the origin with vertex $(4, 0)$ and focus $(6, 0)$ is (A) $20x^2 - 16y^2 = 320$ (B) $16x^2 + 20y^2 = 320$ (C) $x^2 + y^2 = 160$ (D) $18x^2 - 20y^2 = 300$ (E) $20x^2 - 16y^2 = 400$

SOLUTION

vertices $= (4, 0) = (a, 0)$ and focus $= (6, 0) = (c, 0)$ then its type A hyperbola

$a = 4 \quad c = 6$

$c^2 = a^2 + b^2$ or $b^2 = c^2 - a^2$

$b^2 = 6^2 - 4^2 = 36 - 16 = 20$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad b > 1$$

$$\frac{x^2}{4^2} - \frac{y^2}{20} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$

multiply through 320 (L.c.m)

$20x^2 - 16y^2 = 320$ A

$20x^2 - 16y^2 = 320$

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TRUTH: The thoughts of the wicked are abomination to the Lord but the words of the pure are pleasant words (Proverb 15:26)

RATE AND PROPORTION

Rate has to do with change in amount, size or dimension of an object with respect to time

$$R = \frac{d\bar{m}}{dt} \qquad m = \text{material/object}$$

EXAMPLE

A pump is inflating a spherical ball whose radius at a certain instant is 2m and is increasing at a rate of 0.01m/s (1) At what rate is the pump working
(2) If air continues to be pumped into the balloon to be increased at this rate, what rate will the radius be increasing be increasing when it is 5m

SOLUTION

For the volume working rate, its a spherical ballon and the volume of sphere $V = \frac{4}{3}\pi r^3$

differentiate the volume with respect to radius

33

$$\frac{dV}{dr} = 3 \times \frac{4}{3}\pi r^2 = 4\pi r^2$$

at an instant when $r = 2m$

$$\frac{dV}{dr} = 4\pi(2)^2 = 16\pi m^2$$

From the question, increasing rate of radius $\frac{dr}{dt} = 0.01m/s$

$$\text{Volume working rate} = \frac{dV}{dt} = \frac{dV}{\cancel{dr}} \times \frac{\cancel{dr}}{dt}$$

$$= 16\pi m^2 \times 0.01m/s$$

$$= 0.16\pi m^3/s$$

(11) What is the increasing radius rate at $r = 5m$?

$$\frac{dr}{dt} = \frac{dr}{\cancel{dV}} \times \frac{\cancel{dV}}{dt} \quad \text{where}$$

$$\frac{dV}{dr} = 4\pi r^2 \text{ then } \frac{dr}{dV} = \frac{1}{4\pi r^2}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \times 0.16\pi m^3/s$$

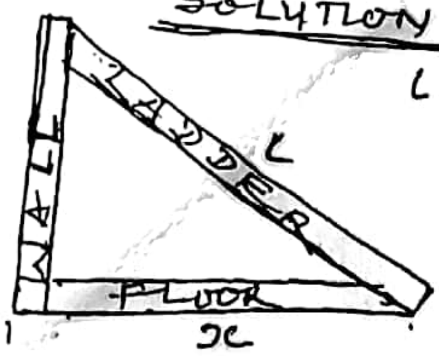
$$\frac{dr}{dt} = \frac{1}{4\cancel{\pi}(5)^2 m^2} \times 0.16\cancel{\pi} m^3/s$$

$$\frac{dr}{dt} = \frac{0.16}{100} m/s = 0.0016 m/s$$

EXAMPLE

The top of the ladder 50cm long is resting against a vertical wall standing on a horizontal floor. Suppose the foot of the ladder is to be pulled away from the wall at the rate of 3cm/min
1) How far is the top of the ladder descending when the foot is 14cm from the wall?
2) How far is the top of the ladder descending when the top and the foot of the ladder moves at the same rate?
3) How far is the top of the ladder descending when the top of the ladder descended at the rate of 4cm/min?

SOLUTION



L = length of the ladder
L = 50

By pythagoras $y^2 + x^2 = L^2$

$$y^2 + x^2 = 50^2 \quad \text{---- eq(1)}$$

1) When the foot is 14cm from the wall

$$x = 14cm$$

$$y^2 + 14^2 = 50^2$$

$$y^2 = 50^2 - 14^2$$

$y = \sqrt{2500-196} = \sqrt{2304} = 48cm$

from eqn (1),

$x^2+y^2 = 50^2$ --- eqn(1)

differentiate with respect to time (t)

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \text{ --- eq(ii)}$$

the rate of pulling the foot $= \frac{dx}{dt} = 3cm/min$

the foot is 14cm away from the floor

$x = 14cm$

Therefore $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

$$14cm \times \frac{3cm}{min} + y\frac{dy}{dt} = 0$$

$$y\frac{dy}{dt} = \frac{-14\times3\,cm^2}{min}$$

when $y = 48cm$

$$48cm \times \frac{dy}{dt} = \frac{-14\times3cm^2}{min}$$

$$\frac{dy}{dt} = \frac{-14\times3cm^2}{48cm\times min} = -\frac{7}{8}cm/min$$

the top of the ladder is descending at the rate of $-\frac{7}{8}cm/min$ and its 48cm far

(2) (34)

The top and the foot of the ladder (which are moving in opposite direction) will moves at the same rate when $\frac{dx}{dt} = -\frac{dy}{dt}$

the negative sign shows that they are moving in opposite direction

From eqn (ii)

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$x\left(-\frac{dy}{dt}\right) + y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt}(-x+y) = 0,$$

$-x+y = 0/\frac{dy}{dt}$ or $-x+y=0$

$\boxed{x = y}$

from eqn (1)

$x^2+y^2 = 50^2$

$y^2+y^2 = 50^2$ $2y^2 = 2500$

$y^2 = 1250$ $y = \sqrt{1250} = 35.36cm$

The top of the ladder is descending at the rate of $-\frac{7}{8}cm/min$ and its ~~35.3~~ 35.36cm far

(3)

Now if the rate of descending of the top of the ladder $\frac{dy}{dt} = -4cm/min$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

negative sign is for descending

and the rate of pulling the foot $\frac{dx}{dy} = 3cm/min$

$$x(3) + y(-4) = 0$$

$$3x - 4y = 0$$

$$y = \frac{3}{4}x \text{ or } x = \frac{4}{3}y$$

from eqn (1)

$$x^2+y^2 = 50^2$$

$$\left(\frac{4y}{3}\right)^2 + y^2 = 50^2$$

$$\frac{16y^2}{9} + \frac{y^2}{1} = 2500$$

$$16y^2 + 9y^2 = 9\times2500$$

$$25y^2 = 9\times2500$$

$$y^2 = \frac{9\times2500}{25}$$

$$y^2 = 900$$

$$y = \sqrt{900}$$

$$y = 30cm$$

The top of the ladder is descending at the rate of $-4cm/min$ and its 30cm far

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...: A true witness delivereth souls ... a deceitful witness speaketh lies (Proverb 14:25)

...VED PAST QUESTIONS AND ANSWERS

...012: The luminous intensity I candelas of a lamp is given by $I = 6\times10^{-4}V^2$ where V is the voltage

8) The rate of change of luminous with voltage when V = 200volts is
(A) 24cd/V (B) 0.24cd/V (C) 0.024cd/V (D) 240cd/V (E) Undefined

SOLUTION

rate of change of the luminous intensity is with respect to voltage not time

$$I = 6\times10^{-4}V^2$$

differentiate with respect to voltage.

$$\frac{dI}{dV} = 12\times10^{-4}V \quad \text{at } V=200$$

$$\frac{dI}{dV} = 12\times10^{-4}\times200$$

$$= 0.24cd/V \qquad (B)$$

9) The voltage at which the light is increasing at a rate of 0.3 candelas per volt is (A) 200V (B) 154V (C) 100V (D) 186V (E) 250V

SOLUTION

$$\frac{dI}{dV} = 0.3cd/V$$

$$\frac{dI}{dV} = 12\times10^{-4}V$$

$$0.3cd/V = 12\times10^{-4}V \qquad (E)$$

$$V = \frac{0.3cd/V}{12\times10^{-4}} = 250V$$

2009: No3: A snowball is increasing in volume at the rate of $10cm^3/hr$. How fast is the surface area increasing when the radius of the ball is 5cm? (A) $4cm^2/hr$ (B) $8cm^2/hr$ (C) $\frac{\pi}{10}cm^2/hr$ (D) $40cm^2/hr$ (E) None of the above

SOLUTION

Snowball is spherical in shape
area of sphere $A = \underline{4\pi r^2}$

Volume of a sphere $= \frac{4}{3}\pi r^3$

If $V = \frac{4}{3}\pi r^3$, differentiate w.r.t "r"

$$\frac{dV}{dr} = 3\times\frac{4}{3}\pi r^2 = 4\pi r^2 \qquad \boxed{r=5cm}$$

$$\frac{dV}{dr} = A = 4\pi r^2 = 4\pi(5)^2$$

$$\frac{dV}{dr} = 100\pi m^2 \text{ and } \frac{dV}{dt} = 10cm^3/hr$$

$$\frac{dr}{dt} = \frac{dV}{dt}\times\frac{dr}{dV} \qquad \boxed{\frac{dr}{dV} = \frac{1}{\frac{dV}{dr}} = \frac{1}{100\pi}}$$

$$\frac{dr}{dt} = 10cm^3/hr \times \frac{1}{100\pi cm^2} = \frac{1}{10\pi}cm/hr$$

$$A = 4\pi r^2$$

$$\frac{dA}{dr} = 8\pi r \quad \text{at } r=5cm$$

$$\frac{dA}{dr} = 8\pi\times5 = 40\pi cm \qquad \frac{dr}{dt} = \frac{1}{10\pi}cm/hr$$

$$\frac{dA}{dt} = \frac{dA}{dr}\times\frac{dr}{dt} = 40\pi cm\times\frac{1}{10\pi}cm/hr$$

$$\frac{dA}{dt} = 4cm^2/hr \qquad (A)$$

34) The population (y) of Bacteria in a colony grows exponentially according to the equation $\frac{dy}{dt} = Ky$ where K is constant and t is measured in years. If the population doubles every 10 years then the value of K is
(A) 0.069 (B) 0.200 (C) 0.301 (D) 3.22 (E) 5.00

SOLUTION

$$\frac{dy}{dt} = Ky$$

$$\frac{dy}{y} = Kdt$$

Integrate both side
as the population double every year

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$$\int_1^2 \frac{dy}{y} = \int_0^{10} k\,dt$$

Population double every year $y_1 = 1$ $y_2 = 2$

$$\ln y\Big]_1^2 = kt$$

$t = 10yr$

$$\ln 2 - \ln 1 = 10k$$

$$0.69 - 0 = 10k \qquad k = \frac{0.69}{10}$$

$$k = 0.069 \quad (A)$$

40) The rate of decomposition of a chemical is proportional to the quantity present. If 100grams reduces to 50gram in 2 hours. The amount remaining after 3hours (to the nearest whole number) is (A) 20gm (B) 25gm (C) 35gm (D) 50gm (E) 125gm

Solution

Method I

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = kN$$

$$\frac{dN}{N} = k\,dt$$

The sample reduces from 100grams to 50g i.e $N_1 = 100$ $N_2 = 50$

Integrate both side

$N_1 \to N_2$
$100 \quad 50$

$$\int_{50}^{100} \frac{dN}{N} = \int_0^2 k\,dt \qquad t = 2$$

$$\boxed{\ln N\Big]_{50}^{100} = kt\Big]_0^2}$$

$$\ln 100 - \ln 50 = 2k \quad (36)$$

$\boxed{N_1 \to N_2 \to N_3 \quad 100 \to 50 \to}$

$$4.605 - 3.912 = 2k$$

$$2k = 0.693$$

$$k = \frac{0.693}{2} = 0.3465$$

The amount remain after 3hours

$N_2 \to N_3$

$$\boxed{\ln N\Big]_{N_3}^{50} = kt\Big]_2^3}$$

$$\ln 50 - \ln N_3 = 3k - 2t$$

$$\ln 50 - \ln N_3 = k$$

$$3.912 - \ln N_3 = 0.3465$$

$$\ln N_3 = 3.912 - 0.3465$$

$$\ln N_3 = 3.5655 \quad (C)$$

$$N_3 = e^{3.5655} = 35gm$$

Method II

The time taken for a sample of 100g to reduce to half of its original amount (50g) = 2hrs i.e half life = 2hrs

$$N = N_0 e^{-\lambda t} \qquad t_{1/2} = 2hrs \quad N_0 = 100gram$$

$$\lambda = \frac{0.693}{t_{1/2}} = \frac{0.693}{2} = 0.3465 \qquad \boxed{t = 3hrs}$$

$$N = 100\,e^{-0.345\times 3}$$

$$N = 100\,e^{-1.035} = 100 \times 0.354$$

$$N \simeq 35gm \quad (C)$$

Method III

$$\frac{N}{N_0} = \frac{1}{2^k} \quad \text{or} \quad N = \frac{N_0}{2^k}$$

$$k = t/t_{1/2} = 3/2 = 1.5 \quad (C)$$

$$N = \frac{100}{2^{1.5}} = \frac{100gm}{2.828} = 35gm$$

2009) No 21: The rate of decomposition of a chemical is proportional to the quantity present. If 200grams reduces to 50grams in 1hour. How much will remain after 3hours (A) 6.250gm (B) 3.125gm (C) 25gm (C) 50gm (E) 125gm

SOLUTION

$$\ln N\Big]_{50}^{200} = kt\Big]_0^1 \qquad \text{at } t = 1hr$$

$$\ln 200 - \ln 50 = k \times 1$$

$$5.298 - 3.912 = k$$

$$k = 1.3863$$

$$\ln N\Big]_{N_3}^{50} = kt\Big]_1^3 \qquad \text{at } t = 3hr$$

$$\ln 50 - \ln N_3 = 3k - 1k$$

$$\ln 50 - \ln N_3 = 2k$$

$$\ln N_3 = \ln 50 - 2k$$

$$\ln N_3 = 3.912 - 2(1.3863)$$

$$\ln N_3 = 1.1394 \quad (B)$$

$$N_3 = e^{1.1394}$$

$$= 3.1248 = 3.125mg$$

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TRUTH: Fools make a mock at sin but among the righteous there is favor (Proverbs 14:9)

ERROR AND APPROXIMATION

The difference between approximated value and actual value is called error. That is

Error = Actual Value − Approximated Value

$$\% \text{Error} = \frac{\text{Error}}{\text{Actual Value}} \times 100\%$$

The approximation of numerical values lead to truncation error

EXAMPLE

If $z = 2\sqrt{9+x^2}$. find the approximate change in z when x is decreased from 4 to 3.99

SOLUTION

$$z = 2\sqrt{9+x^2}$$

$$z = 2(9+x^2)^{1/2}$$

Find change in z w.r.t x

$$\frac{\Delta z}{\Delta x} = \frac{1}{\cancel{2}} \times \cancel{2}(9+x^2)^{-1/2} \times 2x$$

$$\frac{\Delta z}{\Delta x} = \frac{2x}{(9+x^2)^{1/2}}$$

37

$$\Delta z = \frac{2x}{\sqrt{9+x^2}} \times \Delta x$$

$$\Delta x = x_2 - x_1 = 3.99 - 4 = -0.01$$

$$\Delta z = \frac{2 \times 4}{\sqrt{9+4^2}} \times -0.01 \quad \text{at } x = 4$$

$$\Delta z = \frac{8}{\sqrt{9+16}} \times -0.01 = \frac{8}{\sqrt{25}} \times -0.01$$

$$\Delta z = \frac{8}{5} \times -0.01 = -0.016$$

Approximate change in $z = -0.016$

EXAMPLE

Find the approximate change in volume of a cube of side $x$cm given by increasing the side by 5%

SOLUTION

$$\text{Volume of cube} = l^3 = x^3$$

$$\frac{dV}{dx} = 3x^2 \qquad dx = 5\%$$

$$dV = 3x^2 dx \qquad dx = 0.05$$

$$dV = 3x^2 \times 0.05 = 0.15x^2$$

EXAMPLE

Use differential approximate to find the value of (A) $\sqrt[4]{17}$ (B) $\sqrt{1020}$ (C) $\cos 59$ (D) $\tan 44$

SOLUTION

$$\sqrt[4]{17} = \sqrt[4]{16+1}$$

$x = 16$ and $\Delta x = 1$

let $y = \sqrt[4]{x}$

$$y = (x)^{1/4}$$

Find change in y with respect to x

$$\frac{\Delta y}{\Delta x} = \frac{1}{4} x^{-3/4} = \frac{1}{4x^{3/4}}$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{4(16)^{3/4}} = \frac{1}{4 \times 8} = \frac{1}{32}$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{32} \quad \text{or} \quad \Delta y = \frac{1}{32} \times \Delta x$$

$$\Delta y = \frac{1}{32} \times 1 = \frac{1}{32}$$

$$\Delta y = y_1 - y$$

$$y_1 = \Delta y + y \quad \text{or}$$

$$y_1 = y + \Delta y$$

$$y_1 = \sqrt[4]{x} + \Delta y$$

$$y_1 = \sqrt[4]{16} + \frac{1}{32}$$

$$y_1 = 2 + \frac{1}{32}$$

$$y_1 = 2 + 0.03125$$

$$y_1 = 2.03125$$

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(B) $\sqrt{1020} = \sqrt{1024+(-4)}$

$x = 1024 \quad \Delta x = -4$

Let $y = \sqrt{x}$

$y = x^{1/2}$

$\frac{\Delta y}{\Delta x} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

$\frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{1024}} = \frac{1}{2\times 32}$

$\frac{\Delta y}{\Delta x} = \frac{1}{64}$ or $\Delta y = \frac{1}{64} \times \Delta x$

$\Delta y = \frac{1}{64} \times -4 = \frac{-1}{16}$

$y_1 = y + \Delta y = \sqrt{x} + \Delta y$

$y_1 = \sqrt{1024} + \left(\frac{-1}{16}\right)$

$y_1 = 32 - 0.0625$

$= 31.9375$

(C) $\cos 59 = \cos(60 + (-1))$

$x = 60^\circ \quad \Delta x = -1^\circ$

Let $y = \cos x$

$\frac{\Delta y}{\Delta x} = -\sin x$

$\frac{\Delta y}{\Delta x} = -\sin 60^\circ = -0.8660$

$\boxed{180^\circ = \pi}$

$1^\circ = \frac{\pi}{180}$

$\boxed{-1^\circ = \frac{-\pi}{180} = \frac{-3.142}{180}}$

$\Delta y = -0.8660 \times \Delta x$

$\Delta y = -0.8660 \times \frac{-3.142}{180}$

$\Delta y = 0.01512$

$y_1 = y + \Delta y$

$y_1 = \cos 60 + \Delta y$

$= 0.5000 + 0.01512$

$= 0.5152$

truncation error or Error due to approximation $= 0.5152 - 0.5150$

$= 0.0002$

NOTE: Cos59 in Calculator $= 0.5150$

(D) $\tan 44^\circ = \tan[45^\circ + (-1^\circ)]$

$x = 45^\circ \quad \Delta x = -1^\circ$

Let $y = \tan x$

$\frac{\Delta y}{\Delta x} = \sec^2 x$

$\boxed{\sec x = \frac{1}{\cos x}}$

$\frac{\Delta y}{\Delta x} = (\sec 45^\circ)^2$

$\frac{\Delta y}{\Delta x} = \left(\frac{1}{\cos 45}\right)^2 = \left(\frac{1}{1/\sqrt{2}}\right)^2 = 2$

$\Delta y = 2 \times \Delta x = 2 \times \frac{-3.142}{180}$

$\Delta y = -0.0349$

$f(x + \Delta x) = f(x) + \Delta y$

$= \tan x + (-0.0349)$

$= \tan 45 - 0.0349$

$= 1 - 0.0349$

$= 0.9651$

NOTE: $\tan 44^\circ$ in calculator $= 0.9657$

truncation error $= 0.9657 - 0.9651$

$= 0.0006 = 0.06\%$

$= 0.0006 \times 100\% = 0.06\%$

SOLVED PAST QUESTIONS & ANSWERS

2015: NO 27: The breadth of a rectangle is 3cm if the breath increases by 0.1cm, given that the perimeter of the rectangle is 8cm then the new area of the rectangle is (A) $2.1cm^2$ (B) $3.3cm^2$ (C) $2.8cm^2$ (D) $4.2cm^2$ (E) none

SOLUTION

perimeter $= 2(L+B) = 8cm$ then

or $L + B = 4$ (divide by 2)

$L = 4 - B$ --- eqn (1)

Area $= L \times B$

$A = (4-B)B$

$A = 4B - B^2$

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TRUTH? He that walketh with wise men shall be wise but a companion of fools shall be destroyed (Proverb 13:20)

$$A = 4B - B^2$$

$$\frac{\Delta A}{\Delta B} = 4 - 2B \quad \text{at } B = 3cm$$

$\Delta B = 0.1cm$

$$\frac{\Delta A}{\Delta B} = 4 - 2(3)$$

$$\frac{\Delta A}{\Delta B} = -2 \text{ or } \Delta A = -2\Delta B$$

$$\Delta A = -2 \times 0.1 = -0.2cm^2$$

Since $L = 4 - B$ and $B = 3$

$$L = 4 - 3 = 1$$

$$A = L \times B = 1 \times 3 = 3cm^2$$

New Area = Initial area + change in Area

$$A_1 = A + \Delta A$$

$$A_1 = L \times B + \Delta A$$

$$A_1 = 3 \times 1 + (-0.2)$$

$$A_1 = 3 - 0.2 = 2.8cm^2$$

C

(29) If $Z = \sqrt[3]{x^2 - 1}$, the approximate change in Z, where x increases from 4 to 4.01 is (A) 4.0101 (B) 0.0420 (C) 0.0420 (D) 0.0400 (E) 0.0044

SOLUTION

39

$$Z = \sqrt[3]{x^2-1}$$

$$Z = (x^2-1)^{1/3}$$

$$\frac{\Delta Z}{\Delta x} = \frac{1}{3} \times (x^2-1)^{-2/3} \times 2x$$

$$\frac{\Delta Z}{\Delta x} = \frac{1 \times 2x}{3 \times (x^2-1)^{2/3}}$$

$$\Delta x = 4.01 - 4 = 0.01$$

$$\Delta Z = \frac{2x}{3(x^2-1)^{2/3}} \times \Delta x \quad \text{at } x = 4$$

$$\Delta Z = \frac{2(4)}{3(4^2-1)^{2/3}} \times 0.01$$

E

$$\Delta Z = \frac{8}{3(15)^{2/3}} \times 0.01 = 0.0044$$

2013 No 15: Using differential, the approximate value of $\sqrt{49.5}$ to four decimal places is (A) 7.0000 (B) 0.1357 (C) 7.0357 (D) 7.1357 (E) 7.3571

SOLUTION

$$\sqrt{49.5} = \sqrt{49 + 0.5}$$

$$x = 49 \quad \Delta x = 0.5$$

Let $y = \sqrt{x}$

$$y = x^{1/2} \quad \frac{\Delta y}{\Delta x} = \frac{1}{2}x^{-1/2}$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}} \quad \Delta y = \frac{1}{2\sqrt{x}} \times \Delta x$$

$$\Delta y = \frac{1}{2\sqrt{49}} \times 0.5 = \frac{1}{2 \times 7} \times 0.5 = \frac{0.5}{14}$$

$$y_1 = y + \Delta y$$

$$y_1 = \sqrt{x} + \Delta y$$

$$y_1 = \sqrt{49} + \Delta y$$

C

$$y_1 = 7 + \frac{0.5}{14} = 7 + 0.0357$$

$$y_1 = 7.0357$$

NOTE; press $\sqrt{49.5}$ in Calculator = 7.0356

In Examination/Test use your Calculator the nearest number to 7.0356 is C

2012: No 33 The change in y if x changes from 2.50 to 2.51 when $y = 2x - x^2$ is (A) 0.03 (B) −0.03 (C) 0.02 (D) −0.02 (E) 0.51

SOLUTION

$$y = 2x - x^2$$

$$\frac{\Delta y}{\Delta x} = 2 - 2x$$

$$\Delta x = 2.51 - 2.50 = 0.01$$

$$\Delta y = (2 - 2x) \times \Delta x \quad \text{at } x = 2.50$$

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$\Delta y = 2 - 2(2.50) \times 0.01$ (B)

$\Delta y = (2 - 5) \times 0.01 = -3 \times 0.01 = -0.03$

(35) The time of swing T of a pendulum is given by $T = K\sqrt{L}$ where K is constant the percentage error in time of swing if the length changes from 32.1cm to 32.0cm is (A) 0.156% (B) 0.165% (C) -0.165% (D) -0.156% (E) 0.01

SOLUTION

$T = K\sqrt{L}$

since K is constant let $K = 1$

$T = \sqrt{L}$ or $T = L^{1/2}$

$\frac{\Delta T}{\Delta L} = \frac{1}{2} L^{-1/2} = \frac{1}{2\sqrt{L}}$

$\Delta L = 32 - 32.1 = -0.1cm$

$\Delta T = \frac{1}{2\sqrt{L}} \times \Delta L$ at $L = 32.1cm$

$\Delta T = \frac{1}{2\sqrt{32.1}} \times -0.1$ $\boxed{T = \sqrt{L}}$

$\Delta T = -0.0088$

Error $= \Delta T = -0.0088$

% Error $= \frac{\Delta T}{T} \times 100\% = \frac{-0.0088}{\sqrt{L}} \times 100\%$

% Error $= \frac{-0.0088}{\sqrt{32.1}} \times 100\%$ (40)

$= \frac{-0.0088}{5.6657} \times 100\%$ (D)

$= -0.156\%$

2008: No 7: The side of a cube is measured with error of 3%. the corresponding percentage error in the volume of the cube is (A) 9% (B) 6% (C) 3% (D) 0.03% (E) 90%

SOLUTION

$V = L^3$

$\frac{\Delta V}{\Delta L} = 3L^2$

~~Error = 3% of L~~

side of the cube is measure with error of 3% i.e $\Delta L = 3\%$ of $L = 0.03L$

$\Delta V = 3L^2 \times \Delta L = 3L^2 \times 0.03L$

$\Delta V = 0.09L^3$ $\boxed{V = L^3}$

% Error $= \frac{\Delta V}{V} \times 100\%$

$= \frac{0.09L^3}{L^3} \times 100\% = 9\%$ (A)

THE GREATEST THING IN THE WORLD

"for God so loved the world that he gave his only begotten Son that whosoever believeth in him should not perish but have everlasting life (John 3:16).

Love is the implicit thing that makes life worth living. If you are not loving you are not living. We live to love and we love to live. People dont care how much we know until they know how much we care. We can never know how better we can live until we learn how better we can love. You might have really prayed but if you have not really loved you have not really lived. The greatest relationship is to love God and be loved by God. It is better not to live than not to love.

"He that hate his brother is a murderer".

Love is the fulfilment of holy commandment. He that has no reason to love has no reason to live. Every other things would fail but love can never fail. Love is the energy of life and essence of living. If you are not loving, you are not living. Love is eternal. Let us love one another because God is love

TEXT

1 Corithian 13: 1 - 10

1 John 2, 3, 4

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TRUTH: A false balance is an abomination to the Lord but a just weight is his delight (Proverb 11:1)

SOLUTION OF NON-LINEAR ALGEBRAIC EQUATION

NEWTON'S METHOD

Newton's method is one of the methods that use to find the roots of non-linear algebraic equation. It is given as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where $n = 0, 1, 2, 3 - - -$

ADVANTAGES OF NEWTON'S METHOD	DISADVANTAGES OF NEWTON'S METHOD
(i) Its simple	(i) It requires good Initial guess
(ii) It has fast convergence	(ii) It requires symbolic evaluation such as $f'(x)$ and $f(x)$
(iii) It can be easily extended to multidimensional systems or higher order equations	

NOTE: $f(x)$ means function of x while $f'(x)$ means derivative (differential) of function of x

SOLVED QUESTIONS AND ANSWERS

2013: No 23: An approximation to the root of the equation $x^3 - x - 5 = 0$ near the point $x = 1.9$ is (A) $x_1 = 1.904$ (B) $x_1 = 1.896$ (C) $x_1 = 1.984$ (D) $x_1 = 1.804$ (E) $x_1 = 1.994$

SOLUTION

$$x^3 - x - 5 = 0$$

Let $f(x) = x^3 - x - 5$

$$f'(x) = \frac{d f(x)}{dx} = 3x^2 - 1$$

at Initial guess $x_0 = 1.9$

$$f(x_0) = f(1.9) = 1.9^3 - 1.9 - 5$$

$$f(x_0) = -0.041$$

$$f'(x_0) = 3(1.9)^2 - 1 = 9.83$$

By Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

when $n = 0$

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x)}{f'(x_0)}$$

A

$$x_1 = 1.9 - \frac{(-0.041)}{9.83}$$

$$= 1.9 + 0.004 = 1.904$$

2012 No 38: If $x_0 = 2.000$ is the Initial approximation to the real root of the equation $x^3 - 5x + 3 = 0$ Using Newton's method, a better approximation to the root is (A) 2.142 (B) 1.857 (C) 2.100 (D) 2.500 (E) None of the above

SOLUTION

$$x^3 - 5x + 3 = 0$$

Let $f(x) = x^3 - 5x + 3$

$$f'(x) = \frac{d f(x)}{dx} = 3x^2 - 5$$

$$f(x_0) = f(2.000) = 2^3 - 5(2) + 3 = 1$$

$$f'(x_0) = f'(2) = 3(2)^2 - 5 = 7$$

when $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{1}{7} = 2 - 0.1429$$

$$= 1.857 \quad \text{B}$$

2011 No 36 If $x_0 = 1.500$ is the initial approximation to the real root of the equation $x^3 - 3x + 1 = 0$ Using Newton method, a better approximation to the root is (A) 1.166 (B) -5.000 (C) 0.834 (D) 1.533 (E) -0.375

SOLUTION

$$f(x) = x^3 - 3x + 1$$

$$f'(x) = 3x^2 - 3$$

$$f(x_0) = f(1.5) = 1.5^3 - 3(1.5) + 1$$
$$= -0.125$$

$$f'(x) = f'(1.5) = 3(1.5)^2 - 3 = 3.75$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{(-0.125)}{3.75}$$

$$x_1 = 1.5 + 0.0333 = 1.533 \quad D$$

2010: No 15: If $x_0 = 2.000$ is the initial approximation to the root of the equation $x^3 - 3x - 4 = 0$ Using Newton's method, a better approximation root is (A) 2.142 (B) 1.857 (C) 2.222 (D) 2.500 (E) None of the above

SOLUTION

$$f(x) = x^3 - 3x - 4$$

$$f'(x) = 3x^2 - 3$$

$$f(x_0) = f(2) = 2^3 - 3(2) - 4 = -2$$

$$f'(x_0) = f'(2) = 3(2)^2 - 3 = 9$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{-2}{9}$$

$$x_1 = 2 + \frac{2}{9} = 2 + 0.222$$

$$x_1 = 2.222 \quad C$$

2015: No 15: Using Newton-Raphson method the approximate root of the equation $2x^3 - 5x^2 + 2x + 1 = 0$ if $x_0 = 1.2$ is (A) 1.21 (B) 1.3 (C) 0.87 (D) 0.95 (E) NONE

SOLUTION

$$f(x) = 2x^3 - 5x^2 + 2x + 1$$

$$f'(x) = 6x^2 - 10x + 2$$

$$f(x_0) = f(1.2) = 2(1.2)^3 - 5(1.2)^2 + 2(1.2) + 1$$
$$= 3.456 - 7.2 + 2.4 + 1$$
$$= -0.344$$

$$f'(x_0) = f'(1.2) = 6(1.2)^2 - 10(1.2) + 2$$
$$= 8.64 - 12 + 2 = -1.36$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.2 - \frac{(-0.344)}{(-1.36)}$$

$$x_1 = 1.2 - 0.2529 = 0.95 \quad D$$

SEVEN FACTS ABOUT FAILURE

1) He that fail to learn will learn to fail
2) Failure is cumulative not accidental
3) Failure is a choice not a chance
4) Excuse is the modern language of failure
5) Procrastination is the pathway to failure
6) The only thing a man can achieve in life without effort is failure
7) Laziness and slothfulness are the parent of failure

SEVEN KEYS TO FAILURE (7 "T"s)

I) Trying to please everybody
II) Trying to be another person
III) Trying to be independent of God
IV) Trying to be independent of others
V) Trampling on divine opportunity
VI) Turning back during time of trial
VII) Trading what you need most in life for immediate needs.

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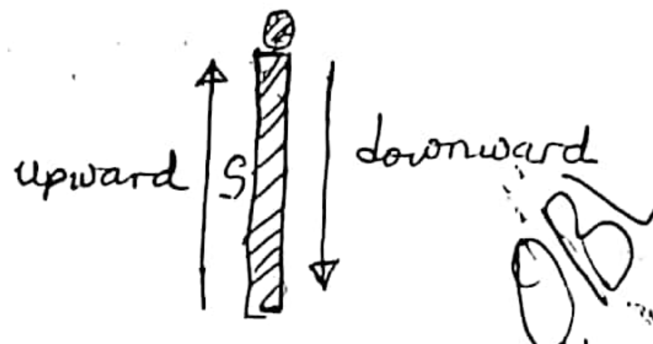
TRUTH! The fear of the Lord prolongeth days but the years of the wicked shall be shortened (Proverb 10:27)

41 ## MOTION UNDER GRVITY

It has been experimentally proved that near the earth suface, falling object has a constant acceleration due to gravity of $9.8m/sec^2$ or $32ft/sec^2$ provided air resistance is neglected



If a rolling metal ball fall through a distance (S) from the top of a cliff, the constant acceleration $a = +32ft/sec^2$ or $a = 9.8m/s^2$ (This is for free falling object of downward direction)

If the rolling ball moves in upward direction against the force of gravity, the constant acceleration is $a = -32ft/sec^2$ or $a = 9.8m/s^2$

43

Suppose we have

$$a = \frac{dv}{dt}$$

$$dv = a\,dt$$

Integrate both side

$$\int dv = \int a\,dt$$

$$V = at + C_1 \text{ --- eqn (1)}$$

also $V = \frac{ds}{dt}$

$$at + C_1 = \frac{ds}{dt}$$

$$ds = (at + C_1)dt$$

Integrate both side

$$S = \frac{at^2}{2} + C_1 t + C_2 \text{ --- eqn (11)}$$

where C_1 and C_2 are constant of Integration

at intial position, S_0 for Initial Velocity U, the time $t = 0$

$$S_0 = \frac{a(0)^2}{2} + C_1(0) + C_2$$

$$\boxed{C_2 = S_0 \text{ and } C_1 = U}$$

from eqn (1) $V = at + C_1$ at initial velocity U, $t = 0$

$U = a(0) + C_1$ then $C_1 = U$

Substitute for S_0 and U in eqn (11)

$$\boxed{S = \frac{at^2}{2} + Ut + S_0 \text{ --- eqn (111)}}$$

EXAMPLE

A ball is thrown upward reaches the height of $h = 20 + 80t - 16t^2$ at the end of t sec. (1) find the velocity and acceleration when $t = 2$

(11) How long will it take for the ball to rise to the highest point?

(111) What is the highest point attained by the ball?

SOLUTION

(1) $h = 20 + 80t - 16t^2$

differentiate the height or distance with respect to time

$$\frac{dh}{dt} = 80 - 32t$$

$$V = \frac{dh}{dt} = 80 - 32t \quad \text{at } t = 2$$

$$V = 80 - 32(2) = 80 - 64$$

$$V = 16ft/sec$$

(2) for the object to reach the highest point then the

final Velocity $V = 0$

$V = 80 - 32t = 0$

$32t = 80 \quad t = \frac{80}{32} = 2\frac{1}{2}$ sec

The ball reach the highest point at $2\frac{1}{2}$ sec

(3) The highest point is at $2\frac{1}{2}$ sec

$h = 20 + 80t - 16t^2$ at $t = 2\frac{1}{2} = \frac{5}{2}$

$h = 20 + 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2$

$= 20 + 200 - 100 = 120$ ft

SOLVED PAST QUESTIONS AND ANSWERS

2015 NO 24 If $S(t) = -16t^2 + 100t + 400$ represents the position of a moving object, find its velocity at $t = 0$

(a) 500 (b) 200 (c) 300 (d) 400 (e) 100

SOLUTION

$S(t) = -16t^2 + 100t + 400$

differentiate the distance with respect to time

$\frac{ds}{dt} = -32t + 100t^0$

$V = \frac{ds}{dt} = -32t + 100$ at $t = 0$

$V = -32(0) + 100 = 100$ ft/sec

E

30) A point moves so that its displacement is given by $S = 16 + 27t - t^3$ when and where will it stop its direction of motion? (A) $t = 3$, $S = 70$ (b) $t = 70$, $s = 3$ (c) $t = 4$, $S = 144$ (d) $t = 5$, $S = 144$ (e) none

SOLUTION

"When" or "how long" has to do with time

"Where or "how far" has to do with distance

The object will stop when it reaches its highest point at which the final velocity becomes zero

$S = 16 + 27t - t^3$

differentiate with respect to time

$\frac{ds}{dt} = 0 + 27 - 3t^2$

$V = 27 - 3t^2$ (at highest point $V = 0$)

$V = 27 - 3t^2 = 0$ then $3t^2 = 27$ or

$t^2 = \frac{27}{3}$, $t^2 = 9$ $t = \sqrt{9} = 3$ sec

The highest point or maximum displacement is at 3 sec

A

$S = 16 + 27t - t^3$ at $t = 3$

$S = 16 + 27(3) - 3^3 = 16 + 81 - 27 = 70$ feet

2013 NO 32. If $S(t) = -16t^2 + 100t + 400$ represents the position of a moving object find its initial velocity

(A) 500 (B) 200 (C) 300 (D) 400 (E) 100

SOLUTION

$S(t) = -16t^2 + 100t + 400$

$S(t) = \frac{at^2}{2} + Ut + S_0$ --- eq (iii)

by comparison $\frac{a}{2} = -16$ $a = -32$ ft/s2

$U = 100$ ft/s, $S_0 = 400$ ft

Initial velocity $U = 100$ ft/s

E

(34) If the height of a ball thrown upward is given by by $H = 20 + 80t - 16t^2$ at any time t, its velocity at $t = 2$ is (a) 16 (b) 32 (c) 20 (d) 80 (e) 64

SOLUTION

$H = 20 + 80t - 16t^2$

differentiate with respect to time

$\frac{dH}{dt} = 80 - 32t$ at $t = 2$

$V = \frac{dH}{dt} = 80 - 32(2)$

$V = 80 - 64$

$V = 16$ feet/sec

A

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TRUTH: When pride cometh, then cometh shame but with the lowly is wisdom (Proverb 11:2)

SOLVED PAST QUESTIONS AND ANSWERS

2012: NO 37: If the velocity of a particle moving along the x-axis is $v(t) = 2t - 4$ and its position is 4, then at any time t, its position $S(t)$ is (A) $t^2 - 4t + 4$ (B) $t^2 - 4t - 4$ (C) $t^2 + 4t + 4$ (D) $t^2 - 4t$ (E) $2t + 4$

SOLUTION

$V = \frac{ds}{dt}$ $\quad V = 2t - 4$

$2t - 4 = \frac{ds}{dt}$ or $(2t-4)dt = ds$

then $ds = (2t-4)dt$

Integrate both side w.r.t time

$S = \frac{2t^2}{2} - 4t + C_1$

45

$S = t^2 - 4t + C_1$

at initial position $S_0 = 4$, $t = 0$

$4 = 0^2 - 4(0) + C_1$

$4 = 0 - 0 + C_1$ $\quad C_1 = 4$ then

$\boxed{S = t^2 - 4t + 4}$ (A)

2011: NO 10: The distance moved by a car in time t seconds is given by $x = 3t^3 - 2t^2 + 4t - 1$, the velocity and acceleration at $t = 0$ are (A) $V = 4m/s$ & $A = 0m/s^2$ (B) $V = 7m/s$ & $A = -4m/s^2$ (C) $V = 4m/s$ & $A = -4m/s^2$ (D) $V = A = -4m/s$ (E) $V = A = 4m/s$

SOLUTION

$x = 3t^3 - 2t^2 + 4t - 1$

differentiate with respect to time

$V = \frac{dx}{dt} = 9t^2 - 4t + 4$ at $t = 0$

$V = 9(0)^2 - 4(0) + 4 = 4m/s$

$V = 9t^2 - 4t + 4$

differentiate with respect to time

$A = \frac{dV}{dt} = 18t - 4$ at $t = 0$

$A = 18(0) - 4 = -4m/s^2$ (C)

12) The distance covered by a fallen object is described as $x = \frac{1}{2}gt^2$, $g = 9.8m/s^2$, determine the velocity and acceleration after fallen for 2 seconds (A) $V = 19.6m/s$ & $A = 9.8m/s^2$ (B) $V = 19.6m/s$ & $A = 10m/s^2$ (C) $V = A = 19.6m/s$ (D) $V = 20m/s$ & $A = 9.6m/s^2$ (E) $V = A = 9.6m/s$

SOLUTION

$x = \frac{1}{2}gt^2$

differentiate with respect to time

$\frac{dx}{dt} = 2 \times \frac{1}{2}gt = gt$ at $t = 2$

$V = \frac{dx}{dt} = gt = 9.8 \times 2 = 19.6m/s$

$V = gt$

differentiate with respect to time

$A = \frac{dv}{dt} = g = 9.8m/s^2$ (A)

20) The distance travelled by a vehicle in time t seconds after the brakes were applied is $x = 20t - \frac{5}{3}t^2$ then the speed of the vehicle at $t = 0$ in km/hr (A) 36km/hr (B) 20km/hr (C) 72km/hr (D) 12km/hr (E) 5.56km/hr

SOLUTION

$x = 20t - \frac{5}{3}t^2$

differentiate with respect to time

$V = \frac{dx}{dt} = 20 - \frac{10t}{3}$ at $t = 0$

$V = 20 - \frac{10}{3}(0) = 20km/hr$ (B)

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36) A missile fired from the ground rises x metres vertically upwards in t seconds and $x = 100t - \frac{25}{2}t^2$. The initial velocity is (A) 100m/s (B) 50m/s (C) 45m/s (D) 125m/s (E) 25m/s

SOLUTION

$x = 100t - \frac{25t^2}{2}$

at initial position (x_0), $t=0$, $V=0$

$x_0 = 0$

then $x = -\frac{25t^2}{2} + 100t + 0$

$S = \frac{at^2}{2} + ut + S_0$

By comparison $\frac{a}{2} = -\frac{25}{2}$ $a = -25 ft/s$

$u = 100m/s$ $S_0 = 0$ (A)

37) A rocket projected vertically upward from the ground with an initial velocity of 40m/s. The maximum height attained by the rocket is (A) 100m (B) 80m (C) 160m (D) 40m (E) None of the above [$g = 10m/s^2$]

46

SOLUTION

$S = \frac{at^2}{2} + ut + S_0$ $u = 40m/s$, $S_0 = 0$, $a = -10m/s^2$ [upward against gravity]

$S = -\frac{10t^2}{2} + 40t + 0$

$\boxed{S = -5t^2 + 40t}$

$V = \frac{ds}{dt} = -10t + 40$

at maximum height final velocity = 0

$V = -10t + 40 = 0$ or $10t = 40$ $t = 4$

$S = -5t^2 + 40t = -5(4)^2 + 40(4)$ (B)

$S = -80 + 160 = 80m$

2009 No.9: A rocket is fired (straight up) into the air with initial velocity of 80m/s. How far has it travelled in 2 seconds (A) 120m (B) 140m (C) 180m (D) 100m (E) 160m

SOLUTION

$S = \frac{at^2}{2} + ut + S_0$

$S_0 = 0$, $u = 80m/s$ $t = 2sec$

$a = -g = -10m/s^2$ (upward against gravity)

$S = -\frac{10}{2}(2)^2 + 80(2) + 0$ (B)

$S = -5(2)^2 + 160 = -20 + 160 = 140m$

24) A particle moving along a straight line has its position at any time t, given $S = 12t - t^3$ where S is in metres and t in seconds. At what time is the particle at rest? (A) 2 seconds (B) 4 seconds (C) 3 seconds (D) 5 seconds (E) 1 second

SOLUTION

If the particle start from rest $u = 0$

If the particle is already moving but its later brought to rest $V = 0$

$S = 12t - t^3$

$V = \frac{ds}{dt} = 12 - 3t^2$ $\boxed{V=0}$

$V = 12 - 3t^2 = 0$ or $3t^2 = 12$ (A)

$t^2 = 12/3$, $t^2 = 4$, $t = \sqrt{4} = 2$ seconds

38) The velocity of a particle moving is $V = 5 - 3t$. If its initial position is 4m what is its position after 2 seconds? (A) 10m (B) 8m (C) 12m (D) 2m (E) 5m

SOLUTION

$\frac{ds}{dt} = V$ $ds = Vdt$ $ds = (5-3t)dt$

Integrate with respect to time

$S = 5t - \frac{3t^2}{2} + C_1$ $\boxed{C_1 = S_0 = 4m}$

$S = 5t - \frac{3t^2}{2} + 4$ at $t = 2sec$

$S = 5(2) - \frac{3}{2}(2)^2 + 4 = 10 - 6 + 4$

$S = 14 - 6 = 8m$ (B)

...: An hypocrite with his mouth destroyeth his neighbor but through knowledge shall the just be delivered (Proverb 11 : 9)

MINIMUM AND MAXIMUM POINTS AND VALUE

2009 No 29: The function defined by $f(x) = x^3 - 3x^2$ for all real numbers x has a relative maximum when x (A) -2 (B) 0 (C) 1 (D) 2 (E) 4

SOLUTION

$$f(x) = x^3 - 3x^2$$

$$\frac{df(x)}{dx} = 3x^2 - 6x$$

at critical point $\frac{df(x)}{dx} = 0$

$$3x^2 - 6x = 0$$
$$x(3x - 6) = 0$$
$$x = 0 \text{ or } 3x - 6 = 0$$
$$x = 0 \text{ or } 3x = 6$$
$$x = 0 \text{ or } x = 6/3 = 2$$

NOTE: The smallest value of x gives the maximum value while the minimum value is obtained at highest value of x i.e $x = 2$ (D)

No 32: The function defined by $f(x) = 3x^3 - 6x^2 + 3$, has a relative maximum when x is (A) -12 (B) 0 (C) 1 (D) $2/3$ (E) $4/3$

SOLUTION

$$f(x) = 3x^3 - 6x^2 + 3$$

$$\frac{df(x)}{dx} = 9x^2 - 12x$$

at critical point $\frac{df(x)}{dx} = 0$

$$9x^2 - 12x = 0$$
$$x(9x - 12) = 0$$
$$x = 0 \text{ or } 9x - 12 = 0$$
$$x = 0 \text{ or } x = 12/9$$
$$x = 0 \text{ or } x = 4/3$$

Minimum value is at $x = 4/3$
Maximum value is at $x = 0$ (B)

2012: The turning points of curve $y = x^3 - 3x + 5$ are (A) $(7, 1)$ and $(3, 1)$ (B) $(1, 7)$ and $(1, 3)$ (C) $(-1, 7)$ and $(1, 3)$ (D) $(1, 3)$ and $(-1, 5)$ (E) $(0, 3)$ and $(0, 7)$

SOLUTION

$$y = x^3 - 3x + 5$$

$$\frac{dy}{dx} = 3x^2 - 3$$

at critical point $\frac{dy}{dx} = 0$

$$3x^2 - 3 = 0$$
$$3(x^2 - 1) = 0$$

divide through by 3

$$x^2 - 1 = 0 \text{ or } x^2 - 1^2 = 0$$
$$x^2 - 1^2 = (x - 1)(x + 1) = 0$$
$$x - 1 = 0 \text{ or } x + 1 = 0$$
$$x = +1 \text{ or } -1$$

Minimum value is at $x = +1$

$$y = x^3 - 3x + 5$$
$$y = 1^3 - 3(1) + 5 = 1 - 3 + 5 = 3$$

Minimum point $= (x, y) = (1, 3)$

Maximum value is at $x = -1$

$$y = (-1)^3 - 3(-1) + 5 = -1 + 3 + 5 = 7$$

Maximum point $= (x, y) = (-1, 7)$

turning point = Maximum point and Minimum point $= (-1, 7)$ & $(1, 3)$ (C)

Turning point is also called stationary point

2) Suppose the gradient before and after a point P on $y = f(x)$ is such that its both increasing then P is called (A) maximum point (B) minimum point (C) point of inflection (D) Minimum or maximum point (E) None of the above

SOLUTION

Answer

2013: The minimum point of the function $y = \frac{(x-1)^2}{x}$ is (A) $(1,1)$ (B) $(1,-1)$ (C) $(-1,0)$ (D) $(0,1)$ (E) $(1,0)$

SOLUTION

$$y = \frac{(x-1)^2}{x} = \frac{x^2-2x+1}{x} = x-2+\frac{1}{x}$$

$$\frac{dy}{dx} = 1-0-\frac{1}{x^2} = 1-\frac{1}{x^2}$$

at critical point $\frac{dy}{dx} = 0$

$$1-\frac{1}{x^2} = 0$$

$$1 = \frac{1}{x^2} \text{ or } x^2 = 1$$

$$x = \pm\sqrt{1} = +1 \text{ or } -1$$

Minimum value is at $x = +1$

$$y = x-2+\frac{1}{x} = 1-2+1/1$$

$$y = 1-2+1 = 0$$

Minimum point $(x,y) = (1,0)$ (E)

2012: No 33. If the graph of $y = x^3+ax^2+bx-4$ has a point of inflexion at $(1,-6)$ what is the value of b (A) -3 (B) 0 (C) 1 (D) 3 (E) -1

SOLUTION

$$y = x^3+ax^2+bx-4 \quad (1,-6)$$

$$-6 = 1^3+a(1)^2+b(1)-4$$

$$-6 = 1+a+b-4$$

48

$$a+b = -6+4-1$$

$$a+b = -3 \text{ --- eqn (i)}$$

$$y = x^3+ax^2+bx-4$$

$$\frac{dy}{dx} = 3x^2+2ax+b$$

at point of inflexion $\frac{dy}{dx} = 0$

$$3x^2+2ax+b = 0$$

at $x = 1$

$$3(1)^2+2a(1)+b = 0$$

$$3+2a+b = 0$$

$$2a+b = -3 \text{ --- eqn (ii)}$$

~~Sol~~ eqn (i) and eqn (ii) are given

$$a+b = -3 \text{ --- eqn (i)}$$

$$2a+b = -3 \text{ --- eqn (ii)}$$

Subtract eq (i) from (ii)

$$2a-a = -3 - -3$$

$$2a = -3+3$$

$$2a = 0 \quad a = 0 \text{ when}$$

$$a+b = -3$$

$$0+b = -3 \quad b = -3$$

(A)

2015 No 20: The stationary point of the curve $y = x^3-3x^2+3$ are (A) $(0,3)$ and $(2,7)$ (B) $(0,-3)$ and $(-2,7)$ (C) $(2,3)$ and $(0,7)$ (D) $(3,0)$ and $(7,2)$ (E) None

SOLUTION

$$y = x^3-3x^2+3$$

$$\frac{dy}{dx} = 3x^2-6x$$

at critical point $\frac{dy}{dx} = 0$

$$3x^2-6x = 0$$

$$x(3x-6) = 0$$

$$x = 0 \text{ or } 3x-6 = 0$$

$$x = 0 \text{ or } x = 6/3 = 2$$

Minimum value is at $x = 2$

$$y = x^3-3x^2+3$$

$$y = 2^3-3(2)^2+3 = 8-12+3 = -1$$

Minimum point $= (2,-1)$

Maximum value is at $x = 0$

$$y = 0^3-3(0)^2+3$$

$$y = 3$$

Maximum point $= (0,3)$

Stationary point = Maximum point and Minimum point

stationary point = $(0,3)$ and $(2,-1)$ (E)

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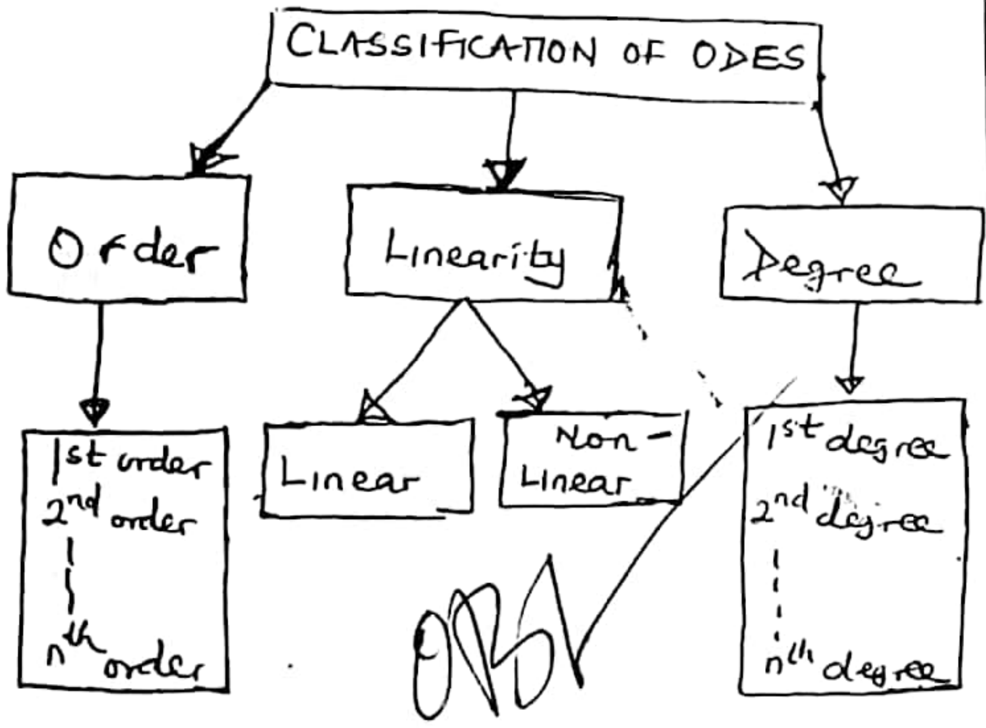
...: He that tilleth his land shall be satisfied with bread (proverb 12:11)

SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

Ordinary Differential Equation (O.D.E) arises from differential equation that has one independent variable which either space (position) or time, is the variable



Order: The order of a differential equation is the order of of the highest derivative present in equation e-g Given an ordinary differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + xy = 0$

49

The highest derivative is the second derivative $\left(\frac{d^2y}{dx^2}\right)$. It's 2nd order equation.

Linearity: A differential equation is non-linear if it contains products of dependent variable or its derivatives or both e.g In any equation where we have

$y \times y = y^2$ $\quad y \times \frac{dy}{dx} = y\frac{dy}{dx}$ or $\frac{dy}{dx} \times \frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2$

such equations are non-linear. If otherwise they are linear

Degree: is the power of the highest derivative in an equation e.g in

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + x^2y = 0$$

The highest derivative is 2nd derivative $\left(\frac{d^2y}{dx^2}\right)$ and its power = 3

The equation is 2nd order and 3rd degree non-linear differential equation

Examples

Examples	Order	Linearity
$\frac{dy}{dx} + y = kx$	1st	Linear
$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = kx$	2nd	non-Linear
$\frac{d^3y}{dx^3} + a\frac{d^2y}{dx^2} + b\left(\frac{dy}{dx}\right)^2 = kx$	3rd	non-linear

The three examples above are all first degree differential equation because the highest derivative in each of them has power of one

EXAMPLE

Form a differential equation by eliminating a constant from the given relation (i) $y = \sqrt{4ax + c}$ (ii) $y = Ax + A^2$ (iii) $y = A\cos x + B\sin x$ (iv) $y^2 = Ax^2 + Bx + C$

SOLUTION

(i) $y = \sqrt{4ax + c}$

Square both side

$y^2 = 4ax + c$

differentiate both sides

NOTE: differentiation of dependent variables (y) attract $\frac{dy}{dx}$ and differentiation of product of two terms involve product rule

$$2y\frac{dy}{dx} = 4a + 0$$

differentiate both side again and apply product rule to L.H.S

$$2\left[y\frac{d^2y}{dx^2} + \frac{dy}{dx} \times \frac{dy}{dx}\right] = 0$$

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divide through by 2

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

At the end we have eliminated Constant a and C.

The equation formed is 2nd order, First degree non-linear differential equation

(II)

$$y = Ax + A^2 \quad --- \text{ eqn } (1)$$

differentiate both sides

$$\frac{dy}{dx} = A + 0 \quad \text{or} \quad A = \frac{dy}{dx}$$

Substitute for A in eqn (1)

$$y = \left(\frac{dy}{dx}\right)x + \left(\frac{dy}{dx}\right)^2$$

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

The equation is first order, 2nd degree linear differential equation.

NOTE: Our aim is to eliminate all the Constant at all cost

50

(III)

$$y = A\cos x + B\sin x \quad --- \text{ eqn } (1)$$

differentiate both sides

$$\frac{dy}{dx} = -A\sin x + B\cos x \quad --- \text{ eqn } (II)$$

differentiate both sides again

$$\frac{d^2y}{dx^2} = -A\cos x + (-B\sin x) - \text{eqn } (III)$$

Add eqn (1) and eqn (III)

$$y + \frac{d^2y}{dx^2} = \cancel{A\cos x} - \cancel{A\cos x} + \cancel{B\sin x} - \cancel{B\sin x}$$

$$y + \frac{d^2y}{dx^2} = 0$$

2nd order, first degree, linear differential equation

(IV)

$$y^2 = Ax^2 + Bx + C$$

differentiate both sides

$$2y\frac{dy}{dx} = 2Ax + B$$

differentiate both sides again and apply product rule to L.H.S

$$\cancel{2}\left[y \times \frac{d^2y}{dx^2} + \frac{dy}{dx} \times \frac{dy}{dx}\right] = \cancel{2}A$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) = A$$

differentiate both sides again and apply product rule to each term in L.H.S

$$\left(y \times \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} \times \frac{dy}{dx}\right) + \left(\frac{dy}{dx} \times \frac{d^2y}{dx^2} + \frac{dy}{dx} \times \frac{d^2y}{dx^2}\right) = 0$$

$$y\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)\frac{dy}{dx} + \frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) + \left(\frac{d^2y}{dx^2}\right)\frac{dy}{dx} = 0$$

$$y\frac{d^3y}{dx^3} + 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) = 0$$

3rd order, first degree, non-linear differential equation

CLASSIFICATION OF DIFFERENTIAL EQUATION

I) Equation with Variable separable
II) Homogeneous equation
III) Exact equation
IV) nth order linear equation
V) Bernoulli equation

VARIABLE SEPERABLE EQUATION

$\frac{dy}{dx} = f(x,y)$ is said to be Variable Separable if it can be expressed in the form $\frac{dy}{dx} = \frac{M(x)}{N(y)}$ or $N(y)dy = M(x)dx$

The solution is

$$\int N(y)dy = \int M(x)dx$$

EXAMPLE

Find the general equation of the differential equation

$$y^2dx - x^2dy = 0$$

SOLUTION

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...H! A man shall be commended ...ording to his wisdom but he that is ...a perverse heart shall be despised (proverb 12:8)

...LVED PAST QUESTIONS AND ANSWERS

2013 NO 33: Each of the following differential equations is linear except (A) $x\frac{dy}{dx} - y = 0$ (B) $y\frac{dy}{dx} - x = 0$ (C) $\frac{dy}{dx} = \frac{y}{x} + 2$ (D) $x^2\frac{dy}{dx} = 0$ (E) $x^2\frac{dy}{dx} = y + 3$

SOLUTION

Answer is B

NO 39: The differential equation $x\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 - 3y = x$ is

(A) first order, second degree, ordinary

(B) second order, second degree, ordinary

(C) Second order, first degree, ordinary

(D) first order, degree one, partial

(E) Second order, second degree, partial

SOLUTION

Answer is (C)

2012 / 40: The following differential equation $\left(\frac{d^3y}{dx^3}\right)^2 + x\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^3 y = x$ is

A) Third order, second degree, ordinary

B) second order, degree three, ordinary

C) Third order, degree three, ordinary

D) ~~First~~ first order, degree degree, ordinary

E) None

SOLUTION

Answer is (A)

2009 NO 20: The general solution of the differential equation $\sin x\frac{dy}{dx} + \cos x y = 1$ is

(A) $y\cos x + x = C$ (B) $y\sin x - x = C$

(C) $y\tan x + x = C$ (D) $y\cot x + 1 = C$

(E) None

SOLUTION

$$\sin x\frac{dy}{dx} + \cos x y = 1$$

divide through by $\sin x$

$$\frac{dy}{dx} + \frac{\cos x}{\sin x}y = \frac{1}{\sin x}$$

$$\frac{dy}{dx} + Py = Q$$

$$I \cdot F = e^{\int P dx} = e^{\int \frac{\cos x}{\sin x}} = e^{\ln \sin x}$$

$$I \cdot F = e^{\ln \sin x} = \sin x.$$

$$y = \frac{1}{I \cdot F}\int Q\,(I \cdot F)\,dx + C$$

$$y = \frac{1}{\sin x}\int \frac{1}{\sin x} \times \sin x\,dx + C$$

$$\sin x \times y = \int \frac{\sin x}{\sin x}dx + C$$

$$y\sin x = \int dx + C$$

$$y\sin x = x + C$$

$$y\sin x - x = C$$

(B)

36) The solution of the differential equation $x\frac{dy}{dx} = 4y$ is? (given that $y = 2$ when $x = -1$) (A) $y(x) = e^{x^2}$ (B) $y(x) = 2x^4$ (C) $y(x) = x^4 + 2$ (D) $y^2 - 2y = 1 - 2x$ (E) None of the above

SOLUTION

$$x\frac{dy}{dx} = 4y$$

$$x\,dy = 4y\,dx$$

$$\frac{1}{4y}dy = \frac{1}{x}dx$$

$$N(y)dy = M(x)dx$$

$$\int N(y)\,dy = \int M(x)dx$$

$$\int \frac{1}{4y}dy = \int \frac{1}{x}dx$$

$$\frac{1}{4}\ln y = \ln x + \ln c$$

$$\ln y^{1/4} = \ln x + \ln c$$

51

Recall $a\log x = \log x^a$ and

$\log A + \log B = \log(A \times B) = \log B$

$\therefore \cancel{\ln} y^{1/4} = \cancel{\ln}(x \times C)$

$$\boxed{y^{1/4} = Cx}$$

Find the Fourth power of both sides

$$(y^{1/4})^4 = (Cx)^4$$

$y = c^4x^4$ at $y = 2$ and $x = -1$

$2 = c^4(-1)^4$

$2 = c^4 \times 1$ then B

$$\boxed{c^4 = 2}$$

If $y = c^4x^4$ then $\underline{y = 2x^4}$

2008: The equation of the curve passing through $(3,2)$ and whose tangent at (x,y) has slope $\frac{x}{y}$ (A) $x^2+y^2=10$ (B) $x^2-y^2=5$ (C) $x^2+y^2=12$ (D) $x^2-y^2=12$ (E) $y=2x+3$

SOLUTION

52

$\frac{dy}{dx}$ = slope/gradient $= m = \frac{x}{y}$

$m = \frac{x}{y}$ at $(3,2)$ $\quad y_1 = 2$, $x_1 = 3$

$m = 3/2$

Equation of tangent to the curve is

$$y - y_1 = m(x - x_1)$$

Equation of normal to the curve is

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

The question is equation of tangent

$y - 2 = \frac{3}{2}(x-3)$

$2(y-2) = 3(x-3)$

$2y - 4 = 3x - 3$

$2y - 3x = -3+4$

$$\boxed{2y - 3x = 1}$$

OR METHOD II

$y - y_1 = m(x - x_1)$ $\quad m = \frac{x}{y}$

$x_1 = 3$, $y_1 = 2$ $\quad (y-2) = \frac{x}{y}(x-3)$

$y(y-2) = x(x-3)$

$y^2 - 2y = x^2 - 3x$

$y^2 - 2y = x^2 - 3x$

$x^2 - y^2 = -2y + 3x$ $\quad$ at $x = 3$, $y = 2$

$x^2 - y^2 = -2(2) + 3(3)$

$x^2 - y^2 = -4 + 9$ $\quad$ B

$x^2 - y^2 = 5$

22) The slope of a curve at any point $P(x,y)$ is always given by the equation $\frac{dy}{dx} = 6x(x+1)$. If the curve passes through $(0,1)$ then the equation of the curve is

(A) $y = 2x^3 + 3x^2 + 1$ (B) $y = 3x^3 + 2x^2$ (C) $y = 2x^3 + 3x^2 - 1$ (D) $y = 6x^2 + 6x + 1$ (E) None of the above

SOLUTION

$m = \frac{dy}{dx} = 6x(x+1)$ at $(0,1)$

~~$m = 6(0)(0+1)$~~

$y - y_1 = m(x - x_1)$

$y - 1 = 6x(x+1)(x-0)$

$y - 1 = 6x(x+1)x$

$y - 1 = 6x^2(x+1)$

$y - 1 = 6x^3 + 6x^2$

$y = 6x^3 + 6x^2 + 1$

OR METHOD II

$\frac{dy}{dx} = 6x^2 + 6x$

Integrate both side

$\int \frac{dy}{dx} = \int 6x^2 + 6x$ at $(0,1)$

$y = 2x^3 + 3x^2 + c$

$1 = 2(0)^3 + 3(0)^2 + c$

$c = 1$ $\quad$ A

$y = 2x^3 + 3x^2 + 1$

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TRUTH: Lying lips are abomination to the Lord but they that deal truly are his delight (Proverb 12:22)

$$y^2dx - x^2dy = 0$$

Collect the like terms

$$y^2dx = x^2dy$$

$$\frac{1}{y^2}dy = \frac{1}{x^2}dx \Rightarrow N(y)dy = M(x)dx$$

Integrate both side

$$\int \frac{1}{y^2}dy = \int \frac{1}{x^2}dx$$

$$\int N(y)dy = \int M(x)dx$$

$$\int y^{-2}dy = \int x^{-2}dx$$

$$\frac{y^{-2+1}}{-2+1} = \frac{x^{-2+1}}{-2+1} + c$$

$$\frac{y^{-1}}{-1} = \frac{x^{-1}}{-1} + c$$

$$\frac{-1}{y} = \frac{-1}{x} + c$$

$$\frac{1}{x} - \frac{1}{y} = c$$

53

FIRST ORDER LINEAR DIFFERENTIAL EQUATION

Given A differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Solution of the equation is given as

$$y = \frac{1}{I \cdot F}\int Q\,(I \cdot F)dx + C$$

where I·F known as integrating factor or integrating function and its given as

$$I \cdot F = e^{\int P dx}$$

EXAMPLE

Solve the differential equation

$$x\frac{dy}{dx} + y = x^4 - 3x$$

SOLUTION

re-write the equation to general form

$$\frac{dy}{dx} + Py = Q$$

$$x\frac{dy}{dx} + y = x^4 - 3x$$

divide through by x

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{x^4 - 3x}{x}$$

$$\frac{dy}{dx} + \frac{1}{x}y = x^3 - 3$$ compared with

$$\frac{dy}{dx} + Py = Q$$ then

$$P = \frac{1}{x} \text{ while } Q = x^3 - 3$$

$$I \cdot F = e^{\int P dx} = e^{\int \frac{1}{x}dx} = e^{\ln x}$$

$$y = \frac{1}{I \cdot F}\int Q\,(I \cdot F)dx + C$$

$$y = \frac{1}{e^{\ln x}}\int (x^3 - 3)e^{\ln x}dx + C$$

$$e^{\ln x}y = \int x^3e^{\ln x} - 3e^{\ln x} + C$$

$$\boxed{e^{\ln x} = x}$$

$$xy = \int x^3 \times x - 3x + C$$

$$xy = \int (x^4 - 3x)dx + C$$

$$\boxed{xy = \frac{x^5}{5} - \frac{3x^2}{2} + C}$$

BERNOULLI EQUATION

NOTE: The other methods are beyond the scope of Math 104

SOLVED PAST QUESTIONS AND ANSWERS

2015: No 12. The order of the differential equation $\left(\frac{d^3y}{dx^3}\right)^4 + 3x\left(\frac{dy}{dx}\right)^5 = 0$ is

(A) 2 (B) 3 (C) 4 (D) 5 (E) NONE

SOLUTION

3rd order, fourth degree non-linear differential equation. (B)

2014: No 3: The associated differential equation of the function $3x^2y^2 + 4y - 4x + 5 = 0$ is

(A) $-\frac{6x^2y + 4}{6x^2y - 4}$ (B) $\frac{6x^2 + 4}{6x^2y - 4}$ (C) $-\frac{6xy^2 - 4}{6x^2y + 4}$

(D) $-\frac{6x^2y + 4}{6x^2y + 4}$ (E) $-\frac{6xy^2 - 4}{6x^2y - 4}$

SOLUTION

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$3x^2y^2+4y-4x+5=0$

differentiate with respect to x

using product rule where there is product of two terms (i.e x^2y^2)

$3[x^2 \times 2y\frac{dy}{dx}+y^2 \times 2x]+4\frac{dy}{dx}-4+0=0$

$3[2x^2y\frac{dy}{dx}+2xy^2]+4\frac{dy}{dx}-4=0$

$6x^2y\frac{dy}{dx}+6xy^2+4\frac{dy}{dx}-4=0$

Collect the like terms

$6x^2y\frac{dy}{dx}+4\frac{dy}{dx}=4-6xy^2$

$\frac{dy}{dx}(6x^2y+4)=-6xy^2+4$

$\frac{dy}{dx}=\frac{-6xy^2+4}{6x^2y+4}$ (D)

19) Which of the following is not a first order, first degree O.D.E (A) Exact D.E (B) Bernoulli equation (C) Homogeneous D.E (D) Variable D.E (E) Linear equation

SOLUTION

Answer is (B)

54

20) The solution of the D.E $(1+y)\frac{dy}{dx}=1+y^2$ is

(A) $y=\tan^{-1}y+\frac{1}{2}\ln(1+y^2)+c$

(B) $y=\tan^{-1}x+\frac{1}{2}\ln(1+x^2)+c$

(C) $x=\tan^{-1}x+\frac{1}{2}\ln(1+y^2)+c$

(D) $x=\tan^{-1}y+\frac{1}{2}\ln(1+y^2)+c$

(E) $x=-\tan^{-1}y+\frac{1}{2}\ln(1+y^2)+c$

SOLUTION

$(1+y)\frac{dy}{dx}=\frac{1+y^2}{1}$

$\frac{1+y}{1+y^2}dy=1\times dx$

Integrate both side

$\int\frac{1+y}{1+y^2}dy=\int dx$

$\int\frac{1}{1+y^2}+\int\frac{y}{1+y^2}=\int dx$

$\tan^{-1}y+\frac{1}{2}\ln(1+y^2)+C=x$ (D)

21) Where P and Q are functions of x only or constant $\frac{dy}{dx}+Py=Q$ is called (A) Exact D.E (B) Bernoulli-type equation (C) Homogeneous equation (D) Variable D.E (E) Linear D.E

SOLUTION

Answer is (E)

22) The question above has an integrating function (A) $e^{\int P(1-n)dx}$ (B) $e^{\int P(n-1)dx}$ (C) $e^{\int pdx}$ (D) $e^{\int pQdx}$ (E) $e^{\int -pdx}$

SOLUTION

Answer is (C)

23) The equation $\cos x\frac{dy}{dx}+y\sin x=1$ has an integrating function (A) $\tan x$ (B) $\sin x$ (C) $\cos x$ (D) $\sec x$ (E) $\operatorname{cosec} x$

SOLUTION

$\cos x\frac{dy}{dx}+y\sin x=1$

divide through $\cos x$

$\frac{dy}{dx}+\frac{\sin x}{\cos x}y=\frac{1}{\cos x}$

$\boxed{e^{\ln a}=a}$

$\frac{dy}{dx}+Py=Q$

$I.F=e^{\int Pdx}=e^{\int\frac{\sin x}{\cos x}dx}=e^{-\ln\cos x}=\frac{1}{e^{\ln\cos x}}$

$I.F=e^{\ln\cos x}=\frac{1}{\cos x}=\sec x$ (D)

2013 No 37: The differential equation formed by eliminating c from $y=cx+1/c$

Solution

$y=cx+\frac{1}{c}$, $\frac{dy}{dx}=c$ then substitute for c in the equation $y=cx+\frac{1}{c}$

$y=\left(\frac{dy}{dx}\right)x+\frac{1}{(dy/dx)}$ $\boxed{\text{then multiply by } dy/dx}$

$y(dy/dx)=\left(\frac{dy}{dx}\right)^2x+1$ (A)

$x\left(\frac{dy}{dx}\right)^2-y\left(\frac{dy}{dx}\right)=-1$

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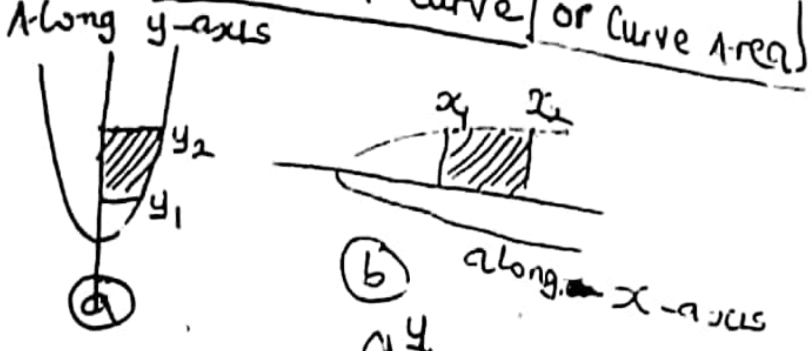
5)

TRUTH: My son, if sinners entice you do not consent (Proverb 1:10)

$\boxed{\text{Area under curve}}$ or Curve Area



Area under curve $= \int_{y_1}^{y_2} x\,dy$ along y-axis

Area under curve $= \int_{x_1}^{x_2} y\,dx$ along x-axis

SOLVED PAST QUESTIONS AND ANSWERS

2015: NO 25 The area of the solid region whose boundaries are $y = 3x^2$, the x-axis $x = 1$ and $x = 4$ is (A) 63 (B) 36 (C) 26 (D) 23 (E) 64

55

SOLUTION

$$\text{Area} = \int_{x_1}^{x_2} y\,dx = \int_1^4 3x^2\,dx = \frac{\cancel{3}x^3}{\cancel{3}}\Big|_1^4$$

$$= x^3\Big|_1^4 = 4^3 - 1^3$$

$$= 64 - 1 = 63 \quad (A)$$

$$= 63 \text{ square unit}$$

2012: NO 4: The area enclosed by the curve $y = 4\cos 3x$, the x-axis and ordinates $x = 0$ and $x = \frac{\pi}{2}$ is (A) $\frac{\pi}{2}$ (B) $\frac{7}{2}$ (C) $2/5$ (D) $\frac{4}{3}$ (E) 3

SOLUTION

$$\text{Area} = \int_{x_1}^{x_2} y\,dx = \int_0^{\pi/2} 4\cos 3x\,dx = \frac{4\sin 3x}{3}\Big|_0^{\pi/2}$$

$$= \frac{4}{3}\sin 3x\Big|_0^{\pi/2} = \frac{4}{3}\left[\sin\left(3\times\frac{\pi}{2}\right) - \sin 0\right]$$

$$= -4/3 \text{ square unit}$$

Area cannot be negative. Take the absolute value (D)

$$A = \frac{4}{3} \text{ square unit}$$

1) The area of the region bounded by the line $y = 1$ and the curve $y = x^2 - 2x + 1$ (in square unit) is (A) $\frac{2}{3}$ (B) 1 (C) $4/3$ (D) 0 (E) $14/3$

SOLUTION

$y = 1$ line

$y = x^2 - 2x + 1 \rightarrow$ Curve

If $y = y'$

then $x^2 - 2x + 1 = 1$ (line & curve)

$$\boxed{x^2 - 2x = 0}$$

$x(x-2) = 0$

$x = 0$ or $x - 2 = 0$

$x = 0$ or $x = 2$

$Y^m = x^2 - 2x$

$$A = \int_{x_1}^{x_2} Y^m\,dx = \int_0^2 (x^2 - 2x)\,dx$$

$$A = \frac{x^3}{3} - \frac{2x^2}{2}\Big|_0^2 = \frac{x^3}{3} - x^2\Big|_0^2$$

$$A = \left[\frac{2^3}{3} - 2^2\right] - \left[\frac{0^3}{3} - 0^2\right]$$

$$\frac{8}{3} - \frac{4}{1} = \frac{8-12}{3} = -\frac{4}{3}$$

$$A = \frac{4}{3} \text{ square unit} \quad (C)$$

2009) No 28. The area of the region bounded by the curve $y = x(2-x)$ and the x-axis (in square units) is (A) $\frac{5}{6}$ (B) $\frac{1}{6}$ (C) $\frac{20}{3}$ (D) $\frac{4}{3}$ (E) $6/5$

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SOLUTION

$y = x(2-x)$

Let $y = 0$, $x(2-x) = 0$

$x(2-x) = 0$

$x = 0$ or $2 - x = 0$

$x = 0$ or $x = 2$

$$\text{Area} = \int_{x_1}^{x_2} y\,dx = \int_0^2 x(2-x) = \int_0^2 (2x - x^2)\,dx$$

$$= \frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\Big|_0^2 = \left[2^2 - \frac{2^3}{3}\right] - 0$$

$$= 4 - \frac{8}{3} = \frac{12-8}{3} = \frac{4}{3} \quad D$$

38) The area of the region bounded by the line $y = 1$ and the curve $y = x^2 - 4x + 4$ (in square units) is (A) $\frac{2}{3}$ (B) -1 (C) $\frac{4}{3}$ (D) 0 (E) $\frac{14}{3}$

SOLUTION

$y = 1 \rightarrow$ line

$y = x^2 - 4x + 4 \rightarrow$ Curve

If $y = y$ then

$x^2 - 4x + 4 = 1$ (line & curve)

$$\boxed{x^2 - 4x + 3 = 0}$$

$(x-1)(x-3) = 0$

$x = 1$ or $x = 3$

$$\text{Area} = \int_{x_1}^{x_2} Y_m\,dx = \int_1^3 (x^2 - 4x + 3)\,dx$$

$$\text{Area} = \frac{x^3}{3} - \frac{4x^2}{2} + 3x\Big|_1^3$$

$$= \left[\frac{3^3}{3} - \frac{4(3)^2}{2} + 3(3)\right] - \left[\frac{1^3}{3} - \frac{4(1)^2}{2} + 3(1)\right]$$

$$= (\cancel{9} - 18 + \cancel{9}) - \left(\frac{1}{3} - 2 + 3\right)$$

$$0 - \left(\frac{1}{3} + \frac{1}{1}\right)$$

$$= -\left(\frac{1+3}{3}\right) = \frac{-4}{3}$$

$= 4/3$ square unit C

56

2008 No 27: The area of the region bounded by the curve $y = x^2$ and the line $y = x$ (in square units) is (A) $\frac{5}{6}$ (B) $\frac{1}{6}$ (C) $2/3$ (D) $5/3$ (E) $6/5$

SOLUTION

$y = x$ line

$y = x^2$ curve

If $y = y$

$x^2 = x$ (line & curve)

$$\boxed{x^2 - x = 0}$$

$x(x-1) = 0$

$x = 0$ or $x = 1$

$$A = \int_{x_1}^{x_2} Y^m\,dx = \int_0^1 (x^2 - x)\,dx = \frac{x^3}{3} - \frac{x^2}{2}\Big|_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{2}\right] - [0] = \frac{2-3}{6} = \frac{-1}{6}$$

Area $= \frac{1}{6}$ square unit B

30) The area of the area of the region enclosed by the graph of $y = x$ and $y = x^2 - 3x + 3$ (in square units) is (A) $\frac{2}{3}$ (B) 1 (C) $\frac{4}{3}$ (D) 0 (E) $14/3$

SOLUTION

$y = x$ line

$y = x^2 - 3x + 3$ curve

If $y = y$

$x^2 - 3x + 3 = x$

$$\boxed{x^2 - 4x + 3 = 0}$$

$(x-3)(x-1) = 0$

$x = 3$ or 1

$$A = \int_{x_1}^{x_2} Y_m\,dx = \int_{x_1}^{x_2} (x^2 - 4x + 3)\,dx = \frac{x^3}{3} - 2x^2 + 3x\Big|_1^3$$

$$= \left[\frac{3^3}{3} - 2(3)^2 + 3(3)\right] - \left[\frac{1^3}{3} - 2(1)^2 + 3(1)\right]$$

$$= 9 - 18 + 9 - \left[\frac{1}{3} - 2 + 3\right]$$

$$= 0 - \left[\frac{1}{3} + 1\right] = -\left(\frac{1+3}{3}\right)$$

$$= -4/3$$

Area $= 4/3$ square unit C

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egion and 's

TRUTH: My son hear the instruction of thy father and forsake not the law of thy mother (Proverb 1:8)

VOLUME OF REVOLUTION

Volume of revolution of a curve is given as

Volume $= \int_{x_1}^{x_2} \pi y^2 dx$ along y-axis or

Volume $= \int_{y_1}^{y_2} \pi x^2 dx$ along x-axis

SOLVED PAST QUESTIONS AND ANSWERS

2015 NO 26: The volume of the solid resulting from revolving the portion of the curve $y = 2 - \frac{x^2}{2}$ from $x=0$ to $x=2$ about the y-axis (A) 4π (B) 3π (C) π (D) 2π (E) None

SOLUTION

$$y = 2 - \frac{x^2}{2}$$

$$y^2 = \left(2 - \frac{x^2}{2}\right)^2 = 4 - 2x^2 + \frac{x^2}{4}$$

$$\text{Volume} = \int_{x_1}^{x_2} \pi y^2 dx = \pi \int_0^2 \left(4 - 2x^2 + \frac{x^2}{4}\right) dx$$

$$= \pi\left(4x - \frac{2}{3}x^3 + \frac{x^3}{12}\right)\Big|_0^2 \quad 57$$

$$= \pi\left(4(2) - \frac{2}{3}\times 2^3 + \frac{2^3}{12}\right) - \pi(0 - 0 + 0)$$

$$= \pi\left(8 - \frac{16}{3} + \frac{8}{12}\right) - 0$$

$$= \pi\left(\frac{8}{1} - \frac{16}{3} + \frac{4}{3}\right)$$

$$= \pi\left(\frac{24 - 16 + 4}{3}\right) \quad A$$

$$= \frac{\pi \times 12}{3} = 4\pi \text{ cubic unit}$$

2009: NO 27: The area between the curve $y = \sin\frac{1}{2}x$, the x-axis and the lines $x=0$ and $x=\pi$ is rotated through 2π radians about the x-axis. The volume of the solid formed is (A) π cu (B) $\frac{\pi}{2}$ cu (C) $\frac{\pi^2}{2}$ cu (D) $\frac{15\pi}{2}$ cu (E) 3π cu

SOLUTION

$$\text{Volume} = \int \pi y^2 dx$$

$$y = \sin\frac{1}{2}x$$

$$y^2 = \sin^2\frac{1}{2}x$$

$$\boxed{\sin^2\frac{1}{2}x = \frac{1 - \cos x}{2}}$$

$$\text{Volume} = \int_0^{\pi} \pi\left(\frac{1 - \cos x}{2}\right) dx \qquad \pi = 180°$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{1 - \cos x}{1} dx$$

$$\frac{\pi}{2}\left[x - \sin x\right]_0^{\pi}$$

$$\text{Volume} = \frac{\pi}{2}\left[[\pi - \sin\pi] - [0 - \sin 0]\right]$$

$$= \frac{\pi}{2}\left[[\pi - \sin 180] - [0 - \sin 0]\right]$$

$$= \frac{\pi}{2}\left[[\pi - 0] - [0 - 0]\right]$$

$$= \frac{\pi}{2} \times \pi = \frac{\pi^2}{2} \text{ cubic unit}$$

SEVEN MYSTERY ABOUT SIN

I) Sin is the ruin and misery of man soul
II) Sin is the most deadly things that kill faster than deadliest disease
III) Sin is rebellion against God
IV) The melody of sin is malady to man soul
V) The pleasure of sin is nightmare
VI) Sin is the worst enemy of man
VII) Sin is a murderer and a destroyer

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AREA UNDER TWO CURVES (PARABOLAS)

Find the area enclosed between the parabolas $y^2 = x$ and $x^2 = y$

SOLUTION

$y^2 = x$

$y_1 = \sqrt{x}$ ---- Curve 1

$y_2 = x^2$ ---- curve 2

If $y_1 = y_2$ then

$x^2 = \sqrt{x}$

square both side

$x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$ or $x^3 - 1 = 0$

$x = 0$ or $x^3 = 1$

$x = 0$ or $x = \sqrt[3]{1} = 1$

$x = 0$ or 1

$$\text{Area} = \int_{x_1}^{x_2} y_1\,dx - \int_{x_1}^{x_2} y_2\,dx$$

$$= \int_0^1 \sqrt{x} - \int_0^1 x^2$$

58

$$= \int_0^1 x^{1/2}dx - \int_0^1 x^2 dx$$

$$= \frac{x^{3/2}}{3/2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1$$

$$= \frac{1^{3/2}}{3/2} - \frac{1^3}{3}$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ square unit}$$

2012:34: The coordinate of the point on the graph $y = 3x^2 - 7x + 2$ where the gradient is -1 (A) $(1, -2)$ (B) $(1, 2)$ (C) $(-1, 2)$ (D) $(1, 1)$ (E) $(-1, -1)$

SOLUTION

$y = 3x^2 - 7x + 2$

$\frac{dy}{dx} = 6x - 7$

where gradient $= \frac{dy}{dx} = -1$

$6x - 7 = -1$

$6x = -1 + 7 = 6$

$x = 6/6 = 1$

$y = 3x^2 - 7x + 2$

$y = 3(1)^2 - 7(1) + 2 = 3 - 7 + 2$

$y = 5 - 7 = -2$

the point $(x, y) = (1, -2)$ (A)

2009 No 2. The equation of the curve passing through $(1, 5)$ and whose ~~gradient~~ tangent line at (x, y) has the slope $4x$ is (A) $y = 2x + 1$ (B) $y = x^2 + 3$ (C) $y = 2x^2 + 3$ (D) $y = 3x^2 - 2$ (E) $y = 2x + 3$

SOLUTION

slope $= \frac{dy}{dx} = 4x$

$dy = 4x\,dx$

Integrate both sides

$\int dy = \int 4x\,dx$

$y = \frac{4x^2}{2} + c$

$y = 2x^2 + c$ at $(1, 5)$

$5 = 2(1)^2 + c$

$5 = 2 + c$

$5 - 2 = c$

$c = 3$

$\boxed{y = 2x^2 + 3}$

Answer is C

TO GOD; THE IMMORTAL, INVISIBLE, MOST PRECIOUS, MOST HOLY AND MOST GLORIOUS MUST BE THE GLORY

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