MATHS 102

Es-Eye-En

not real root — $b^2 < 4ac$

$(2k-1)^2 < 4(k+2)(k+1)$.

$\cancel{4k^2} - 4k + 1 < \cancel{4k^2} + 12k + 8$.

$-4k + 1 < 12k + 8$

or $-4k - 12k < 8 - 1$

$\frac{-16k}{-16} < \frac{7}{-16}$

$k < -7/16$

no real value : $b^2 < 4ac$

real no : $b^2 \geq 4ac$

Math 102. 14/7/2010.

SIGN OF A QUADRATIC FUNCTION.

For some real values of x, a quadratic function may be positive or negative.

Example 1

Consider the quadratic function $y = x^2 - 5x + 6$

$\equiv y = x^2 - 5x + 6$

$\equiv (x-2)(x-3)$.

now y is positive if both factors are positive

i.e $(x-2) > 0$ and $(x-3) > 0$

$\Rightarrow x - 2 > 0$ and $x - 3 > 0$

$x > 2$ and $x > 3$

$\therefore x > 3$

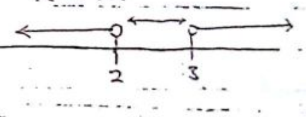
also y is positive if both factors are negative

i.e $(x-2) < 0$ and $(x-3) < 0$

$\Rightarrow x - 2 < 0$ and $x - 3 < 0$

$x < 2$ and $x < 3$

$\therefore x < 2$.



Conclusion

y is negative if x lies between 2 and 3 and positive otherwise.

RATIONAL QUADRATIC FUNCTION.

Rational quadratic function are of the form

$$\frac{ax^2+bx+c}{px^2+qx+r}$$

where a, b, c, p, q, and r are all constant for some real values of x, we can determine the range of values for which the rational function can lie.

Example 2.

Show that if x is real the value of the expression $\frac{(2x+7)(2x-1)}{x-1}$ cannot lie between 8 and 32.

Soln

let $y = \frac{(2x+7)(2x-1)}{x-1}$

$y(x-1) = (2x+7)(2x-1)$

$yx - y = 4x^2+12x-7$

$4x^2+12x-yx-7+y=0$

$4x^2+(12-y)x-7+y=0.$

Condition for real x ($b^2 \geq 4ac$)

where $a=4$, $b=12-y$, $c=y-7$.

$\therefore (12-y)^2 \geq 4\times4(y-7)$

$144-24y+y^2 \geq 16y-112$

Collect like terms:

$y^2-40y+256 \geq 0$

$y^2-8y-32y+256 \geq 0$

$(y^2-8y)(32y-256) \geq 0$

$y(y-8)-32(y-8) \geq 0$

$(y-32)(y-8) \geq 0$

$\therefore$ Case 1:

$(y-32) \geq 0$ and $(y-8) \geq 0$

$y \geq 32$ and $y \geq 8$

$\therefore y \geq 32$.

Case 2:

$(y-32) \leq 0$ and $(y-8) \leq 0$

$y \leq 32$ and $y \leq 8$

$\therefore y \leq 8$

Conclusion: y cannot lies between 8 and 32.

Examples.

If x is real shows that the expression $\frac{(2x-5)(x+1)}{x-1}$ can takes all values.?!

Soln.

let $y = \frac{(2x-5)(x+1)}{x-1}$

$y(x-1) = (2x-5)(x+1)$

$yx - y = 2x^2 - 3x - 5$

$2x^2 - 3x - yx + y - 5 = 0$

$2x^2 - (3+y)x + y - 5 = 0$

for x is real. $b^2 \geq 4ac$.

$(-(3+y))^2 \geq 4 \times 2(y-5)$

$9 + 6y + y^2 \geq 8y - 40$

$y^2 - 2y + 49 \geq 0$

Completing the square method

$y^2 - 2y + 1 + 49 - 1 \geq 0$

$(y-1)^2 + 48 \geq 0$

for all values of y, the expression can take all values.

QED

MAXIMUM & MINIMUM VALUE

Consider $y = ax^2 + bx + c$ ——— ①

$= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$

$= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right]$

$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$

$= a\left[\left(x + b/2a\right)^2 + \frac{4ac - b^2}{4a^2}\right]$

$\therefore a\left(x + b/2a\right)^2 + \frac{4ac - b^2}{4a}$

1. If $a > 0$, then $a(x + b/2a) \geq 0$

$0 \longrightarrow +\infty$

Min value

1. $\Rightarrow$ Minimum value will be when $a(x + b/2a$

either $a \geq 0$ or $(x + b/2a)^2 \geq 0$

either $a = 0$ or $(x + b/2a)^2 = 0$

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but $a > 0$

$(x + b/2a)^2 = 0$

$\therefore x = -b/2a$ and

$y = \frac{4ac - b^2}{4a}$.

2. If $a > 0$, y attained it's minimum values and the point $x = -b/2a$ and $y = \frac{4ac-b^2}{4a}$

i.e. $\left(-b/2a, \frac{4ac-b^2}{4a}\right)$

3. If $a < 0$, then $a(x + b/2a)^2 \leq 0$

$\Rightarrow$ $-\infty$ ← 0

Maximum value will be when $a(x + b/2a)^2 = 0$ either $a = 0$ or $(x + b/2a)^2 = 0$, but $a < 0$

$\Rightarrow (x + b/2a)^2 = 0$

$\therefore x = -b/2a$, $y = \frac{4ac - b^2}{4a}$.

4. If $a < 0$, then y attained it's maximum value at the point

$x = -b/2a$, $y = \frac{4ac - b^2}{4a}$.

i.e $\left(-b/2a, \frac{4ac - b^2}{4a}\right)$

$\Rightarrow$ Math 102 17/7/2010.

Example 1 — Obtained the maximum or minimum from the following:

(i) $12x^2 + 24x + 13$ minimum

(ii) $5 + 6x - x^2$ → max

Soln

1. $12x^2 + 24x + 13$

we have min. value

$x = -b/2a$ $y = \frac{4ac - b^2}{4a}$

$= \frac{-24}{2 \times 12}$ $y = \frac{4 \times 12 \times 13 - (24)^2}{4 \times 12} = 1$

$= -1$

the min pt is $(-1, 1)$

Equation Depending on Quadractic Eqn

Certain Eqn are solved by reducing them to quadractic Eqn.

Eg 2.

Solve the Eqn $x^2+3x-2 = \frac{8}{x^2+3x}$.

Soln

let $y = x^2+3x$.

$\frac{y}{1} - 2 = \frac{8}{y}$

$y^2-2y-8=0$

$y^2-4y+2y-8=0$

$(y^2-4y)(2y-8)=0$.

$y(y-4)+2(y-4)=0$

$(y-4)(y+2)=0$

$y-4=0$ or $y+2=0$

$y=4$ or $y=-2$

where $y=x^2+3x$.

$-2=x^2+3x$ when $y=-2$

$x^2+3x+2=0$

$x^2+2x+x+2=0$

$(x^2+2x)(x+2)=0$

$x(x+2)+1(x+2)=0$

$(x+1)(x+2)=0$

$x+1=0$ or $x+2=0$

$x=-1$ or $x=-2$.

$\Rightarrow$ when $y=4$

$4=x^2+3x$.

$x^2+3x-4=0$.

$x^2+4x-x-4=0$

$(x^2+4x)(x+4)=0$

$x(x+4)-1(x+4)=0$

$(x-1)(x+4)=0$

$x=1$ or $x=-4$.

Eg: 3

Solve the eqn $\sqrt{5x-25}-\sqrt{x-1}=2$.

(Take only +ve values of the sq. root).

Soln

$\sqrt{5x-25}=2+\sqrt{x-1}$

Square both side.

$5x-25=4+4\sqrt{x-1}+x-1$.

$5x - 25 - 4 - x + 1 = 4\sqrt{x-1}$

$4x - 28 = 4\sqrt{x-1}$

$x - 7 = \sqrt{x-1}$

$(x-7)^2 = (\sqrt{x-1})^2$

$x^2 - 14x + 49 = x - 1$

$x^2 - 15x + 50 = 0$

$(x-5)(x-10) = 0$

$x - 5 = 0$ or $x - 10 = 0$

$x = 5$ or $x = 10$

$\Rightarrow$ when $x = 5$

$\sqrt{5x-25} - \sqrt{x-1} = 2$

$\sqrt{25-25} - \sqrt{5-1} = 2$

$-\sqrt{4} \neq 2$.

$\Rightarrow$ When $x = 10$

$\sqrt{50-25} - \sqrt{10-1} = 2$

$\sqrt{25} - \sqrt{9} = 2$

$5 - 3 = 2$.

hence; $x = 10$ is our soln.

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Eg 4.

Find the coordinate of the pt of intersection of Circle $x^2+y^2+6x+4y-13=0$ and $x^2+y^2-10x+10y-15=0$.?

Soln

$x^2+y^2-6x+4y-13=0$ — ①

$x^2+y^2-10x+10y-15=0$ — ②

$4x + 6y + 2 = 0$

$\therefore 2x - 3y + 1 = 0$

$\therefore 2x = 3y - 1$ — ③

$x = \frac{3y-1}{2}$ or

Square $2x = 3y - 1$ both side

$4x^2 = 9y^2 - 6y + 1$ — ④

Multiply Eqn ② by 4:

$4x^2 + 4y^2 - 40y - 40x - 60 = 0$

$9y^2 - 6y + 1 + 4y^2 - 20(3y-1) + 40y - 60 = 0$

$9y^2 + 4y^2 - 6y - 60y + 40y + 1 + 20 - 60 = 0$

$13y^2 - 26y - 39 = 0$

$y^2 - 2y - 3 = 0$.

$(y^2 - 3y)(y - 3) = 0$

$y(y-3)+1(y-3)=0$

$(y+1)(y-3)=0$

$y+1=0$ or $y-3=0$

$y=-1$ or $y=3$

Substitute y into Eqn ③ when $y=-1$

$2x=3y-1$

$2x=3(-1)-1$

$2x=-3-1$

$\frac{2x}{2}=\frac{-4}{2}$, $x=-2$

when $y=3$

$2x=3y-1$

$2x=3(3)-1$

$\frac{2x}{2}=\frac{9-1}{2}$

$x=\frac{8}{2}=4$

$\therefore$ The coordinates are $(4,3)$, $(-2,-1)$

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E.g 5

Shows that if the Eqn $x^2+ax+1=0$ and $x^2+x+b=0$ have common root then $(b-1)^2=(a-1)(1-ab)$

Soln.

Let y be the common root

$y^2+ay+1=0$ —— ①

$y^2+y+b=0$ —— ②

$ay-y+1-b=0$

$y(a-1)+1-b=0$

$y(a-1)=b-1$

$y=\frac{b-1}{a-1}$ —— ③

$\forall a\neq 1$. Substitute Eqn ③ in ①

$y^2+ay+1=0$

$\left(\frac{b-1}{a-1}\right)^2+a\left(\frac{b-1}{a-1}\right)+1=0$

$\frac{(b-1)^2+a(a-1)(b-1)+(a-1)^2}{(a-1)^2}=\frac{0}{1}$

$(b-1)^2+a(a-1)(b-1)+(a-1)^2=0$

$$(b-1)^2+a(ba-a-b+1)+(a^2-2a+1)=0$$
$$(b-1)^2+a^2b-\cancel{a^2}-ab+a+\cancel{a^2}-2a+1=0$$
$$(b-1)^2+a^2b-ab-a+1=0$$
$$(b-1)^2+ab(a-1)-1(a-1)=0.$$
$$(b-1)^2+ab(a-1)-1(a-1)=0$$
$$(b-1)^2+(ab-1)(a-1)=0$$

collect like tem.

$$(b-1)^2=-(ab-1)(a-1).$$
$$(b-1)^2=(a-1)(1-ab)$$

hence the prove.

SQUARE ROOT OF IRRATIONAL QUANTITY:

Suppose we use to find the square root of $a+\sqrt{b}$ (irrational Quantity) we can say that

$$\sqrt{a+\sqrt{b}}=\pm(\sqrt{x}+\sqrt{y}).$$

Note;

$$\sqrt{a+\sqrt{b}}=\pm(\sqrt{x}+\sqrt{y}).$$

square both side

$$\left(\sqrt{a+\sqrt{b}}\right)^2=(\sqrt{x}+\sqrt{y})^2$$

$$a+\sqrt{b}=(\sqrt{x}+\sqrt{y})(\sqrt{x}+\sqrt{y}).$$
$$a+\sqrt{b}=x+\sqrt{xy}+\sqrt{xy}+y.$$
$$a+\sqrt{b}=x+y+2\sqrt{xy}.$$

$\therefore$ by comparing:

$$a=x+y \quad \text{——} (i)$$
$$\sqrt{b}=2\sqrt{xy} \quad \text{——} (ii) \text{ or}$$
$$b=4xy.$$

Example 1.

Find the square root of (i) $6+2\sqrt{5}$ (ii) $18-12\sqrt{2}$.

Soln

1. $6+2\sqrt{5}$. let be $\sqrt{6+2\sqrt{5}}=\pm(\sqrt{x}+\sqrt{y})$.

$$\left(\sqrt{6+2\sqrt{5}}\right)^2=(\sqrt{x}+\sqrt{y})^2.$$
$$6+2\sqrt{5}=(\sqrt{x}+\sqrt{y})(\sqrt{x}+\sqrt{y}).$$
$$6+2\sqrt{5}=x+y+2\sqrt{xy}$$

Compare.

$$6=x+y \quad \text{——} (i)$$
$$2\sqrt{5}=2\sqrt{xy}$$
$$5=xy \quad \text{——} (ii)$$

from (ii) $x=\frac{5}{y}$. —— (iii)

(iii) into (i)

$6 = x + y \Rightarrow 6 = \frac{5}{y} + y$ (L.cm)

$\frac{6}{1} = \frac{5 + y^2}{y}$ Cross multiply.

$6y = 5 + y^2$

$y^2 - 6y + 5 = 0$

$y^2 - 5y - y + 5 = 0$

$(y^2 - 5y)(y - 5) = 0$

$y(y-5) - 1(y-5) = 0$

$(y-5)(y-1) = 0$

$\therefore y = 5$ or $y = 1$.

from above $6 + 2\sqrt{5} = \pm(\sqrt{x} + \sqrt{y})$

Substitute y into Eqn. (iii) for x.

$x = \frac{5}{y}$ when $y = 1$ $x = 5$

when $y = 5$ $x = 1$.

$\therefore (6 + 2\sqrt{5}) = \pm(\sqrt{5} + \sqrt{1})$

$6 + 2\sqrt{5} = (\sqrt{5} + 1)$ //

ii $18 - 12\sqrt{2}$.

$\sqrt{18 - 12\sqrt{2}} = \pm(\sqrt{x} + \sqrt{y})$

$(\sqrt{18 - 12\sqrt{2}})^2 = (\pm(\sqrt{x} + \sqrt{y}))^2$

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$18 - 12\sqrt{2} = (\sqrt{x} + \sqrt{y})^2$

$18 - 12\sqrt{2} = (\sqrt{x} + \sqrt{y})(\sqrt{x} + \sqrt{y})$

$18 - 12\sqrt{2} = x + y + 2\sqrt{xy}$

$\therefore 18 = x + y$ —— (i)

$-12\sqrt{2} = -2\sqrt{xy}$

$6 \times 2 = xy$

$12 = xy$ —— (ii)

$\therefore x = \frac{12}{y}$ —— (iii)

x into Eqn (i).

$18 = \frac{12}{y} + \frac{y}{1} \Rightarrow 18y = 12 + y^2$

$\therefore y^2 + 18y + 12 = 0$

$6 = \frac{5}{y} + \frac{y}{1}$

$\frac{5 + y^2}{y}$

$\Rightarrow 6 = \frac{5 + y^2}{y}$

$6y = 5 + y^2$

$\Rightarrow y^2 + 5 - 6y$

$\Rightarrow y^2 - 6y + 5 = 0$

Alhamdulillah

Exercise

1. Find the square root of the following
(a) $5+2\sqrt{6}$ (b) $a+b+\sqrt{2ab-b^2}$.

2. Find the rational number A & B such that
$8+\sqrt{2}=(a+b\sqrt{2})(6-\sqrt{2})^2$.

PRINCIPLE OF UNDETERMINED COEFFICIENT.

If two polynomials in x are equal for all values of x, we may then equate the coefficients of like powers of x, that is F
$Ax^3+Bx^2+Cx+D=1+nx-mx^2+kx^3$

from undetermine coeff.
$A=k \quad C=n$
$B=-m \quad D=1$.

Example 2 Determine A & B such that
$x^2-4xy+3y^2-2y-1\equiv(x-3y+a)(x-y+b)$
Soln.
$L.H.S \equiv R.H.S$
expand $(x-3y+a)(x-y+b)$.
$\equiv x^2-xy+bx-3xy+3y^2-3by+ax-ay+ab$
$\equiv x^2-4xy+(b+a)x+3y^2-(3b+a)y+ab$
by comparison
$-2=-(3b+a)$ ——— (1)
$-1=ab$ ——— (2)
$0=a+b$ ——— (3)
from Eqn (iii) $a=-b$.
Substitute a into Eqn (1):
$-2=-(3b+a)$.
$2=(3b-b)$
$\frac{2}{2}=\frac{2b}{2}$ $\quad b=1$ and $a=-1$

Example 3.

$$13 = 6\beta + (-1)^2$$
$$13 = 6\beta + 1$$
$$\beta = \frac{12}{6} = 2$$

$$K = 2AB.$$
$$K = 2(-1)(2) = -4$$
$$M = B^2 = (2)^2 = 4$$

ROOTS OF QUADRATIC EQUATION.

If α and β are roots of an equation $ax^2 + bx + c = 0$ then the value of $\alpha^2 + \beta^2$ can be found without first finding the value of α and β. This is done by finding the values of $\alpha + \beta$ and $\alpha\beta$ i.e. the sum of root and the product of root respectively and expressing $\alpha^2 + \beta^2$ or any arbitrary combination of the root in terms of $\alpha + \beta$ & $\alpha\beta$, now if α, β are roots of Eqn $ax^2 + bx + c = 0$ then:

$$ax^2 + bx + c = 0$$
$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}$$

$\alpha + \beta = -b/a$
$\alpha\beta = c/a$

We can write an equation whose roots are α and β in the form;

$$x = \alpha \quad , \quad x = \beta$$
$$x - \alpha = 0 \quad x - \beta = 0$$
$$\Rightarrow (x - \alpha)(x - \beta) = 0$$
$$x^2 - x\beta - x\alpha + \alpha\beta = 0$$
$$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad \text{——} (1)$$

but we says that α and β are roots of the equation $ax^2 + bx + c = 0$, this can be written as;

$$ax^2 + bx + c = 0$$
$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = 0$$
$$\therefore x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{——} (2)$$

Since equation (1) and (2) have the same root, it can be said that they are the same equation written in different ways, so comparing coefficient:

$$-(\alpha + \beta) = b/a \qquad \alpha + \beta = -b/a$$
$$c/a = \alpha\beta \Rightarrow \alpha\beta = c/a.$$

QED

Note: that if we are given root of an [illegible] equation and we are asked to write down the equation, the equation ① gives the equation in it's convinient forms.

Example [illegible]

The equation $4x^2+8x-1=0$, has roots [illegible] find the value of

(i) [illegible] (ii) $(\alpha-\beta)^2$.

Solution:

(i) [illegible] $b=8$, $c=-1$

[illegible] $= -2$ (Sum of roof)

[illegible]

[illegible] (product of root).

[illegible]

But [illegible] occurs in the Xpansion of $(\alpha+\beta)^2$.

$(\alpha+\beta)(\alpha+\beta) = \alpha^2+\alpha\beta+\alpha\beta+\beta^2$

$(\alpha+\beta)^2 = \alpha^2+\beta^2+2\alpha\beta$.

[illegible] $= (\alpha+\beta)^2 - 2\alpha\beta$.

$\frac{[illegible]\beta^2+\alpha^2}{[illegible]\beta^2 [illegible](\alpha\beta)^2} = \frac{\alpha^2+\beta^2}{(\alpha\beta)^2}$

[illegible] substitute the values

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$\alpha^2+\beta^2 - 2\alpha\beta$

$\frac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2} = \frac{(-2)^2-2(-1/4)}{(-1/4)^2}$

$= \frac{4+2/4}{1/16} = 72$.

(ii) $(\alpha-\beta)^2 = (\alpha-\beta)(\alpha-\beta)$.

$\alpha^2-\alpha\beta-\alpha\beta+\beta^2$

$\alpha^2-2\alpha\beta+\beta^2$

$\therefore (\alpha-\beta)^2 = \alpha^2+\beta^2-2\alpha\beta$.

but $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta$.

$(\alpha-\beta)^2 = (\alpha+\beta)^2-2\alpha\beta-2\alpha\beta$.

$(\alpha-\beta)^2 = (\alpha+\beta)^2-4\alpha\beta$.

$\therefore = (-2)^2+4(-1/4)$.

$= 4+5 = 5$

Example 2:

If the root of the equation $2x^2-7x+4=0$ are α and β. Find an equation with integral coefficient whose roots are α/β and β/α.?

Soln

$a=2$, $b=-7$, $c=4$.

(Sum of roof) $\alpha+\beta = -\frac{b}{a} = 7/2$.

Product of root: $\alpha\beta = \frac{c}{a} = \frac{4}{2} = 2$

The Equation

Sum of roof $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$

$\therefore \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \frac{(7/2)^2 - 2(2)}{2}$

$= \frac{49/4 - 4}{2} = \frac{33/4}{2}$

$= \frac{33}{4} \times \frac{1}{2} = \frac{33}{8}$.

Product of roots

$\frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = \frac{\alpha\beta}{\beta\alpha} = \frac{2}{2} = 1$

$\therefore x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$x^2 - \frac{33}{8}x + 1 = 0$

$8x^2 - 33x + 8 = 0$

PARTIAL FRACTION.

When two fractions $\frac{3}{x+2}$ and $\frac{5}{x+1}$ are added together the result is $\frac{8x+13}{(x+2)(x+1)}$

Sometimes it would be required of us to splitt $\frac{8x+13}{(x+2)(x+1)}$ into sum of $\frac{3}{x+2}$ and $\frac{5}{x+1}$.

which is called Partial fraction. The process of splitting and obtained result is called Partial fraction.

Example 1:

Find the value of A, B & C, Such that

$5x+3 \equiv Ax(x+3) + Bx(x-1) + C(x-1)(x+3)$

$5x+3 \equiv (A+B+C)x^2 + (3A-B+2C)x - 3C$

Compared the coefficient

$0 = A+B+C$ —— (i)

$5 = 3A - B + 2C$ —— (ii)

$3 = -3C$ —— (iii)

from equation (3) $3 = -3C$

$C = -\frac{3}{3} = -1$

Substitute $C = -1$ into

$A + B = 1$ —— (4)

$3A - B = 7$ —— (5).

Alternatively, by elimination method.

$5x + 3 \equiv Ax(x+3) + Bx(x-1) + C(x-1)(x+3)$

put $x = -3$.

$-15 + 3 \equiv -3A(-3+3) + 3B(-3-1) + C(-3-1)(-3+3)$

$-15 + 3 \equiv -3B(-3-1)$.

$\frac{-12}{12} \equiv \frac{+12B}{12} \quad B = -1$

put $x = 1$ to eliminate B & C.

$5 + 3 \equiv A(1+3)$

$\frac{8}{4} \equiv \frac{4A}{4} \quad A = 2$

put $x = 0$

$3 \equiv C(-1)(3)$

$\frac{3}{-3} \equiv \frac{C(-3)}{-3} \quad C = -1$

RESULTING PARTIAL FRACTION.

Splitting partial fraction:

(1) Denominator with only linear factors:

Example 1.

Express $\frac{11x+12}{(2x+3)(x+2)(x-3)}$ in p.f.

$\frac{11x+12}{(2x+3)(x+2)(x-3)} \equiv \frac{A}{(2x+3)} + \frac{B}{(x+2)} + \frac{C}{(x-3)}$ L.C.M.

$\frac{11x+12}{\cancel{(2x+3)(x+2)(x-3)}} = \frac{A(x+2)(x-3) + B(2x+3)(x-3) + C(2x+3)(x+2)}{\cancel{(2x+3)(x+2)(x-3)}}$

$11x + 12 \equiv A(x+2)(x-3) + B(2x+3)(x-3) + C(2x+3)(x+2)$

put $x = 3$ to eliminate A & B.

$33 + 12 = C(2(3)+3)(3+2)$

$45 = C(9 \times 5)$.

$\frac{45}{45} = \frac{45C}{45} \quad C = 1$

Put $x = -2$

$-22 + 12 \equiv B(-4+3)(-2-3)$

$-10 = B(-1 \times -5)$

$\frac{-10}{5} = \frac{+5B}{5} \quad B = -2$

Put $x = -3/2$

$-33/2 + 12 = A(-3/2 + 2)(-3/2 - 3)$

$\frac{-33+24}{2} = A\left(\frac{-3+4}{2}\right)\left(\frac{-3-6}{2}\right)$.

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$$\frac{-33+24}{2} = A\left(\frac{11}{2}\right)\left(\frac{-9}{2}\right)$$

$$\frac{-9}{2} = A\left(\frac{-9}{4}\right)$$

$$A = \frac{-9}{2} \times \frac{4}{-9} = \frac{4}{2} \Rightarrow A = 2$$

$$\frac{-11x+12}{(2x+3)(x+2)(x-3)} = \frac{2}{(2x+3)} - \frac{2}{(x+2)} + \frac{1}{(x-3)}$$

(2) Denominator with Quadratic factor:-

Note: fraction which can split into partial fraction are necessarily proper. i.e the degree of the numerator is less than the denominator.

Example 2

Xpress the following partial fractions:

(i) $\frac{1}{x^2+3x+2}$ (ii) $\frac{3x+1}{(x-1)(x^2+1)}$.

Soln

$$\frac{1}{x^2+3x+2} \equiv \frac{1}{(x+1)(x+2)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$$\frac{1}{(x+1)(x+2)} \equiv \frac{A(x+2)+B(x+1)}{\cancel{(x+1)(x+2)}}$$

$$1 \equiv A(x+2) + B(x+1)$$

put $x = -2$

$$1 \equiv A(-2+2) + B(-2+1)$$

$$\frac{1}{-1} = \frac{-B}{-1} \quad B = 1$$

put $x = -1$

$$1 \equiv A(-1+2) + B(-1+1)$$

$$\frac{1}{1} = \frac{A}{1} \quad A = 1$$

$$\therefore \frac{1}{x^2+3x+2} = \frac{1}{(x+1)} + \frac{1}{(x+2)}$$

Note: Generally, the number of constant to be found is the same as the degree of the denominator of the original fraction and If we have a denominator that does not factorize Then the numirator of the partial fraction will be an equation whose degree is less than one, the non-factorizable denominator bearing in mind the number of constant formable

$$\frac{3x+1}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \quad \text{L.c.m.}$$

$$\frac{3x+1}{(x-1)(x^2+1)} \equiv \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$$

$3x+1 \equiv A(x^2+1)+(Bx+C)(x-1)$

$3x+1 = Ax^2 + A + Bx^2 + Bx + Cx - C$

By equating the coefficient

$3x+1 = (A+B)x^2 + (C-B)x + A - C$

$0 = A+B$ — ①

$3 = C-B$ — ②

$1 = A-C$ — ③

from Eqn ② $3 = C-B$ make C the subject

$C = 3+B$ — ④

Substitute into Eqn ③ $1 = A-C$

$1 = A-(3+B)$

$1 = A-3-B$

from eqn ① $A+B=0$ then $A=-B$ — ⑤

$1 = -B-3-B$

$1 = -2B-3$

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$-2B = 1+3$

$\frac{-2B}{-2} = \frac{4}{-2}$ $B = -2$

Substitute B into Eqn ④ $C = 3+B$

$C = 3-2$

from Eqn ⑤ $A = -B$

$A = -(-2)$

$A = 2$

$$\frac{3x+1}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

finally

$$\frac{3x+1}{(x-1)(x^2+1)} = \frac{2}{x-1} + \frac{-2x+1}{x^2+1}$$

Alternatively (2nd Method)

$3x+1 \equiv A(x^2+1)+(Bx+C)(x-1)$

put $x = 1$

$3(1)+1 = A(1)^2+1 + (B(1)+C)(1-1)$

$$3+1 = A(1+1) + (B+c)(0)$$

$$\frac{4}{2} = \frac{2A}{2}$$

$$\therefore A = 2$$

Putting $x = 0$.

$$3x+1 \equiv A(x^2+1) + (Bx+c)(x-1)$$

$$3(0)+1 \equiv A(0^2+1) + (B(0)+c)(0-1)$$

$$\frac{1}{1} \equiv \frac{-c}{-1} \;,\; 1 = A + (-c)$$

$$1 = 2 - c$$

$$\therefore c = +1 \qquad c = -1+2 = 1$$

Putting $x = -1$.

$$3x+1 \equiv A(x^2+1) + (Bx+c)(x-1)$$

$$3(-1)+1 \equiv A((-1)^2+1) + (B(-1)+c)(-1-1)$$

$$-3+1 = 2A + (c-B)(-2)$$

$$-2 = 2A + 2B - 2C$$

$$-2 = 2A + 2B - 2C$$

$$-2 = 2(2) + 2B - 2(1)$$

$$-2 = 4 + 2B - 2$$

$$-4 = 2B$$

$$B = -2$$

$$\frac{3x+1}{(x-1)(x^2+1)} = \frac{2}{(x-1)} + \frac{2x+1}{(x^2+1)}$$

B.) Denominator with repeated factor: Consider the following example

$$\frac{1}{(x-1)^2} \equiv \frac{1}{(x-1)(x-1)} \equiv \frac{1}{x^2-2x+1} = \frac{Ax+B}{x^2-2x+1}$$

$$= \frac{Ax - A + A + B}{x^2-2x+1} = \frac{A(x-1)+A+B}{(x-1)(x-1)}$$

$$= \frac{A\cancel{(x-1)}}{\cancel{(x-1)^2}} + \frac{A+B}{(x-1)^2}$$

$$= \frac{A}{(x-1)} + \frac{A+B}{(x-1)^2}$$

let $B = A+B$.

$$\therefore = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

Note:

1. If $(x^2+1)^2$ is the denominator $\Rightarrow$

$$\frac{1}{(x^2+1)} + \frac{1}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2}$$

In general, if we have $(x-a)^n$ as the denominator it implies, there exist n partial fraction i.e. $\frac{A}{x-a} + \frac{B}{(x-a)^2}$ ...

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$$\cdots + \frac{M}{(x-a)^n}.$$

Example 3.

Express $\frac{1}{(x+2)(x-1)^2}$ into partial fraction

Soln

$$\frac{1}{(x+2)(x-1)^2} \equiv \frac{A}{(x+2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} \quad \text{L.c.m}$$

$$\frac{1}{\cancel{(x+2)(x-1)^2}} \equiv \frac{A(x-1)^2+B(x+2)(x-1)+C(x+2)}{\cancel{(x+2)(x-1)^2}}$$

$$1 \equiv A(x-1)^2+B(x+2)(x-1)+C(x+2).$$

Let $x=1$

$$1 = A(1-1)^2+B(1+2)(1-1)+C(1+2)$$

$$1 = \cancel{A(0)} + \cancel{B(3)(0)} + C(3).$$

$$\frac{1}{3} = \frac{3C}{3} \quad \therefore \quad C = \frac{1}{3}$$

putting $x = -2$

$$1 = A(-2-1)^2 + B(-2+2)(-2-1) + C(-2+2)$$

$$1 = A(-3)^2 + \cancel{B(0)(-3)} + \cancel{C(0)}$$

$$\frac{1}{9} = \frac{9A}{9} \quad \therefore A = \frac{1}{9}.$$

putting $x = -\frac{1}{2}$.

$$1 = A\left(-\tfrac{1}{2}-1\right)^2 + B\left(-\tfrac{1}{2}+2\right)\left(-\tfrac{1}{2}-1\right) + C\left(-\tfrac{1}{2}+2\right).$$

$$1 = A\left(-\tfrac{3}{2}\right)^2 + B\left(\tfrac{3}{4}\right)\left(-\tfrac{3}{2}\right) + C\left(\tfrac{3}{2}\right).$$

$$1 = A\left(\tfrac{9}{4}\right) + B\left(-\tfrac{9}{4}\right) + C\left(\tfrac{3}{2}\right).$$

$$A = \tfrac{1}{9} \;\&\; C = \tfrac{1}{3}.$$

$$1 = \tfrac{1}{9}\left(\tfrac{9}{4}\right) + B\left(-\tfrac{9}{4}\right) + \tfrac{1}{3}\left(\tfrac{3}{2}\right).$$

$$1 = \tfrac{9}{36} - B\left(\tfrac{9}{4}\right) + \tfrac{3}{6}$$

$$1 = \tfrac{1}{4} - \tfrac{9}{4}B + \tfrac{1}{2} \quad \text{L.c.m.}$$

$$1 = \frac{1 - 9B + 2}{4} = \frac{3 - 9B}{4}.$$

$$1 = 1 - \tfrac{9}{4}B \Rightarrow 4 = 3 - 9B$$

$$\therefore \frac{1 - 9B + 2}{4} = \frac{1}{1}$$

$$\therefore 1 - 9B + 2 = 4$$

$$-9B = 4 - 2 - 1$$

$$\frac{\cancel{-9}B}{\cancel{-9}} = \frac{1}{-9} \quad \therefore B = -\frac{1}{9}$$

$$\Rightarrow \frac{1}{(x+2)(x-1)^2} = \frac{1/9}{(x+2)} + \frac{-1/9}{(x-1)} + \frac{1/3}{(x-1)^2}.$$

finally.

$$\frac{1}{(x+2)(x-1)^2} = \frac{1}{9(x+2)} - \frac{1}{9(x-1)} + \frac{1}{3(x+1)}.$$

(4.) Improper fraction: If we have the degree of the numerator is greater that of the denominator, we first divide to obtain the quotient and proper fraction.

Example 4.

Express $\frac{x^4-2x^3-x^2-4x+4}{(x-3)(x^2+1)}$ into partial fraction.

Soln

$$= x+1 + \frac{x^2-2x+7}{(x-3)(x^2+1)}$$

$$= x+1 + \frac{A}{x-3} + \frac{Bx+C}{(x^2+1)}$$

$$= \frac{x+1(x-3)(x^2+1)+A(x^2+1)+(Bx+C)(x-3)}{(x-3)(x^2+1)}.$$

$$\Rightarrow x^4-2x^3-x^2-4x+4 = x+1(x-3)(x^2+1)+A(x^2+1) + Bx+C(x-3).$$

putting $x = 3$

$$(3)^4-2(3)^3-(3)^2-4(3)+4 = 3+1(3-3)(3^2+1)+A(3^2+1) + B(3)+C(3-3).$$

$$81-2(27)-9-12+4 = 2(0)(9+1)+A(9+1)+\cancel{B(3)}+C\cancel{(0)}.$$

$$81-54-9-12+4 = A(10)+3B+C(0).$$

$$10 = 10A$$

$$A = 1.$$

Put $x=0$

$$0^4-2(0)^3-(0)^2-4(0)+4 = 0+1(0-3)(0^2+1)+A(0^2+1) + B(0)+C(0-3)$$

$$4 = -3+A-3C$$

$$4+3-1 = -3C.$$

$$\frac{\cancel{-3}C}{\cancel{-3}} = \frac{6}{-3} \quad C = -2$$

Putting $x = 2.$

$$(2)^4-2(2)^3-(2)^2-4(2)+4 = 2+1(2-3)(2^2+1)+A(2^2+1)+B(2)+C(2-3).$$

$$\cancel{16}-\cancel{16}-4-8+4 = -15+5A+2B-C$$

$$-8 = -15+5A+2B-C.$$

$$2B = -8+15-5(1)+2$$

$$2B = -8+15-5+2 = 0$$

$\frac{2B}{2} = \frac{0}{2}$ $B = 0$

Exercise: Express the ffg into p-fraction

(1) $\frac{x-11}{(x+3)(x-4)}$ (2) $\frac{x}{(25-x^2)}$

(3) $\frac{3x^2-21x+24}{(x+1)(x-2)(x-3)}$ (4) $\frac{5x^2-10x+1}{(x-3)(x^2+4)}$

(5) $\frac{2x^2-x+3}{(x+1)(x^2+2)}$ (6) $\frac{2x^3-x-1}{(x-3)(x^2+1)}$

(7) $\frac{x-5}{(x-2)^2}$ (8) $\frac{x^4+3x-1}{(x+2)(x-1)^2}$

(9) $\frac{x^4+3x-1}{(x+2)(x-1)^2}$ (10) $\frac{5x+4}{(x-1)(x+2)^2}$

(11) $\frac{3x^3+x+1}{(x-2)(x+1)^3}$ (12) $\frac{3x^2+5x-9}{(x^2-1)^2}$

13. $\frac{x^3+2x^2-10x-9}{x^2-9}$.

POLYNOMIALS

A polynomial is the sum of separate terms of positive powers of a variable with a constant term (which may be zero).
The general form of polynomials is
$a_0 + a_1x_1 + \dots + a_{n-1}x^{n-1} + a_nx^n$
$a_1 \dots a_n$
where a_1 to a_n are coefficient in terms to x and a_0 is the constant term.
Note: The highest power of x is called the degree of the polynomials.

ADDITION & SUBTRACTION OF POLYNOMIAL

To add or subtract of polynomials we simplify by simply add or subtract the coefficient of like powers of x.

Example 5

If $P_1 = 2x^3 - 3x^2 + 5x - 7$ and $P_2 = x^3 - x^2 - x + 1$

find (i) $P_1 + P_2$ and $P_1 - P_2$

Soln

(i). $P_1 + P_2 = 2x^3 - 3x^2 + 5x - 7$

$x^3 - x^2 - x + 1$

$3x^3 - 4x^2 - 4x - 6$

$\therefore P_1 + P_2 = 3x^3 - 4x^2 - 4x - 6$

(ii) $P_1 - P_2 = 2x^3 - 3x^2 + 5x - 7$

$x^3 - x^2 - x + 1$

$x^3 - 2x^2 + 6x - 8$

$\therefore P_1 - P_2 = x^3 - 2x^2 + 6x - 8$

MULTIPLICATION OF POLYNOMIAL

Multiplication of polynomial is defined as each term in one polynomial (i.e first polynomial) multiply each term in the second polynomial.

Example 6 If $P_1 = 2x^3 - 3x^2 + 5x - 7$ and $P_2 = x^3 - x^2 - x + 1$, then find

(1) $P_1 \times P_2$

(2) $P_2 \times P_1$

Soln

(1). $P_1 \times P_2 = (2x^3 - 3x^2 + 5x - 7)(x^3 - x^2 - x + 1)$

$= 2x^6 - 2x^5 - 2x^4 + 2x^3 - 3x^5 + 3x^4$

$+ 3x^3 - 3x^2 + 5x^4 - 5x^3 - 5x^2 + 5x - 7x^3 + 7x^2 + 7x$

$= 2x^6 - 2x^5 - 3x^5 - 2x^4 + 3x^4 + 5x^4 + 2x^3 + 3x^3 - 5x^3 - 7x^3$

$-3x^2 - 5x^2 + 7x^2 + 7x + 5x + 7$

$= 2x^6 - 5x^5 + 6x^4 - 7x^3 - x^2 + 12x + 7$

DIVISION OF POLYNOMIALS

A polynomial P_1 can be divided by another polynomial P_2 provided that the degree of P_2 is not greater than P_1. This is done by using the long division method using the x terms as the divisor.

Example 7.

Divide $x^3 + 3x^2 - x + 1$ by $x - 2$.

Soln

$$\begin{array}{r} x^2 + 5x + 9 \\ x - 2 \overline{\smash{)}\, \cancel{x^3} + 3x^2 - x + 1} \\ -\cancel{x^3} \mp 2x^2 \end{array}$$

$$5x^2 - x$$
$$-\underline{5x^2 - 10x}$$
$$9x + 1$$
$$-\underline{9x + 18}$$
$$+19.$$

Therefore +19 is the remainder, $x-2$ is the Divisor and x^2+5x+9 is the Quotient.

We can also Xpress a polynomial in terms of $P = D \times Q + R$

$$x^3+3x^2-x+1 = (x-2)\times(x^2+5x+9)+19.$$

REMAINDER THEOREM

Previously we have seen how a polynomial can be express in terms of the divisor, the Quotient and Remainder. Now the remainder theorem gives a method of finding the remainder without going through the long division process. we know that if we divided;

x^4-5x+6 by $x-2$

we can Xpress the result as $x^4-5x+6 = (x-2)\times$ Quotient + Remainder.

Now if we substitute $x=2$ in the above Xpression, we will have:

$x = 2$

$(2)^4-5(2)+6 = 0\times$ Qoutient + Remainder

$12 =$ Remainder.

Generally, when we divide a polynomial $f(x)$ by a divisor $(x-a)$ the remainder is the value of $f(a)$.

Example 7

Solve the equation $2x^3-3x^2-5x+6=0$ What is the remainder when $f(x)$ is divided by (a) $x-1$ (b) $x+2$ (c) $2x-1$ (d) $3x+2$?

Soln

$f(x) = 2x^3-3x^2-5x+6 = 0$

(a) $x-1=0 \quad x=1$

$f(1) = 2(1)^3-3(1)^2-5(1)+6$

$= 2-3-5+6$

$= 0$

(b) $x+2=0 \quad x=-2$

$f(-2) = 2(-2)^3-3(-2)^2-5(-2)+6$

$= -8 - 12 + 10 + 6$

$= -4$ // Remainder

(c) $2x - 1 = 0$ $x = \frac{1}{2}$.

$f(\frac{1}{2}) = 2(\frac{1}{2})^3 - 3(\frac{1}{2})^2 - 5(\frac{1}{2}) + 6$

$= 2(\frac{1}{8}) - 3(\frac{1}{4}) - \frac{5}{2} + 6$.

$= \frac{2}{8} - \frac{3}{4} - \frac{5}{2} + \frac{6}{1}$

$= \frac{2 - 6 - 20 + 48}{8}$

$= \frac{24}{8} = 3$ // Remainder

(d) $3x + 2 = 0$ $x = -\frac{2}{3}$.

$f(-\frac{2}{3}) = 2(-\frac{2}{3})^3 - 3(-\frac{2}{3})^2 - 5(-\frac{2}{3}) + 6$.

$= 2\left(\frac{-8}{27}\right) - 3\left(\frac{4}{9}\right) + \frac{10}{3} + \frac{6}{1}$

$= \frac{-16}{27} - \frac{12}{9} + \frac{10}{3} + \frac{6}{1}$

$= \frac{-16 - 36 + 90 + 162}{27}$

$= 200$. Remainder.

FACTORS THEOREM.

The remainder theorem can be use to factorize polynomial, when use in this way is called the factor theorem. If $f(x)$ is divided by $(x - a)$, and $f(a) = 0$. Then there is no remainder. This means that $(x - a)$ is the factor of $f(x)$.

Theorem: State that, If $f(a) = 0$, then $x - a$ is a factor of $f(x)$.

Example 8.

Factorize $x^3 - 2x^2 - 5x + 6$?

Soln

$x^3 - 2x^2 - 5x + 6$ — Constant

we know that there will be three linear factors i-e

$f(x) = (x - a)(x - b)(x - c)$ where a, b, c, are positive or negative integers and also multiplying $a \times b \times c$ must give 6 (constant). This implies a, b & c are factors of 6.

By trial and error method.

$\Rightarrow (x \pm 1)$.

$f(x) = x^3 - 2x^2 - 5x + 6$.

$f(-1) = (-1)^3 - 2(-1)^2 - 5(-1) + 6$

$= -1 - 2 + 5 + 6$

$= +8$.

$\therefore (x+1)$ is not a factor.

$f(1) = (1)^3 - 2(1)^2 - 5(1) + 6$.

$= 1 - 2 - 5 + 6$.

$= 0$.

$\therefore (x-1)$ is a factor.

$$\begin{array}{r|l} & x^2 - x - 6 \\ \hline x-1 & x^3 - 2x^2 - 5x + 6 \\ & \underline{x^3 + x^2} \\ & \quad -x^2 - 5x \\ & \quad \underline{+x^2 + x} \\ & \qquad -6x + 6 \\ & \qquad \underline{+6x \pm 6} \\ & \qquad\quad 00 \end{array}$$

The factorize the Quotient $x^2 - x - 6$

$\therefore x^2 + 2x - 3x - 6$

$(x^2 + 2x)(3x + 6)$.

$x(x+2) - 3(x+2)$.

$\therefore (x-3)(x+2)$.

$\therefore$ The factors are $(x-1)(x-3)(x+2)$.

Example 9.

$(x+1)$ is a factor of $a + 3x + 3x^2 + bx^3$ and the remainder when this expression is divided by $x+2$ is 20. find;

a. a and b value

b. Which this values factorize the Xpression.

c. Solve the equation $a + 3x + 3x^2 + bx^3 = 0$.

Soln

a. $(x+1)$ is a factor of $a + 3x + 3x^2 + bx^3$

$f(x) = a + 3x + 3x^2 + bx^3$

$f(-1) = a + 3(-1) + 3(-1)^2 + b(-1)^3$

$\Rightarrow a - 3 + 3 - 1b = 0$

$\Rightarrow a - 1b = 0$

$\therefore a - b = 0$ ——— (i).

$f(-2) = 20$

$= a - 3(-2) + 3(-2)^2 + b(-2)^3$

$\Rightarrow a + 6 + 12 - 8b = 20$

$a - 8b = 20 - 6$

$a - 8b = 14$ —— (ii)

from eqn (i) $a - b = 0$

$\therefore a = b$ —— (iii)

Substitute (a) into Eqn (ii)

$a - 8b = 14$

$b - 8b = 14$

$\frac{-7b}{-7} = \frac{14}{-7}$ $b = -2$

Therefore $a = b$, $a = -2$.

$\therefore a = -2, b = -2$

b. $a + 3x + 3x^2 + bx^3$.

$-2 + 3x + 3x^2 - 2x^3$.

$$
\begin{array}{r|l}
 & -2x^2 + 5x - 2 \\
\hline
x+1 & -2x^3 + 3x^2 + 3x - 2 \\
 & +2x^3 - 2x^2 \\
 & \quad 5x^2 + 3x \\
 & \quad -5x^2 \mp 5x \\
 & \qquad -2x - 2 \\
 & \qquad +2x + 2 \\
 & \qquad 00
\end{array}
$$

$\therefore f(x) = (x+1)(-2x^2 + 5x - 2)$

$= (x+1)(-2x+1)(x-2)$

c. $f(x) = a + 3x + 3x^2 + bx^3 = 0$

$f(x) = (x+1)(-2x+1)(x-2)$

$(x+1) = 0$ $x = -1$

$(-2x+1) = 0$ $x = \frac{1}{2}$

$(x-2) = 0$ $x = 2$

$\therefore x = -1, \frac{1}{2}, 2$

CUBIC AND QUATIC EQUATION

Polynomial of degree three and four is called cubic and Quatic equation(s) respectively.

CUBIC: — Standard form of cubic equation, the general form of cubic equation is

$x^3 + ax^2 + bx^2 + c = 0$ —— *

and the standard form is

$y^3 + Py + q = 0$ —— *

we obtained the standard form by substitute

$y = x + a/3$

This implies

$x = y - a/3$

$x^3 + ax^2 + bx + c = 0$

$\left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c = 0$

Xpand and collect like terms.

$\left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c = 0$

$\Rightarrow y^3 - \frac{3ay^2}{3} + \frac{3ya^2}{9} - \frac{a^3}{27} + ay^2 - \frac{2a^2y}{3} + \frac{a^3}{9} + by - \frac{ab}{3} + c = 0$

$\Rightarrow y^3 - \cancel{ay^2} + \frac{ya^2}{3} - \frac{a^3}{27} + \cancel{ay^2} - \frac{2a^2y}{3} + \frac{a^3}{9} + by - \frac{ab}{3} + c = 0$

$\Rightarrow y^3 + \frac{ya^2}{3} - \frac{a^3}{27} - \frac{2a^2y}{3} + \frac{a^3}{9} + by - \frac{ab}{3} + c = 0$

$\Rightarrow y^3 - \frac{a^2y}{3} - \frac{a^3}{27} + \frac{a^3}{9} + by - \frac{ab}{3} + c = 0$

$y^3 + by - \frac{a^2y}{3} + [illegible] - \frac{ab}{3} + c = 0$

$y^3 + y\left(b - \frac{a^2}{3}\right) + \left([illegible] - \frac{ab}{3} + c\right) = 0$

$y^3 + y\left(b - \frac{a^2}{3}\right) + \left(\frac{3a^3 - a^3 - 9ab + 27c}{27}\right) = 0$

$y^3 + y\left(\frac{3b - a^2}{3}\right) + \left(\frac{2a^3 - 9ab + 27c}{27}\right) = 0$

Compare with the standard equation

$y^3 + py + q = 0$

$p = b - \frac{a^2}{3}$ [illegible]

[illegible]

$q = \frac{2a^3 - 9ab + 27c}{27}$ [illegible]

[illegible]

Write the equation in standard form

$3x^3 + 12x^2 + 18x + [illegible] = 0$

[illegible]

$3x^3 + 12x^2 + 18x + [illegible] = 0$

Divide through by 3

$\frac{3x^3}{3} + \frac{12x^2}{3} + [illegible] = 0$

$x^3 + 4x^2 + \frac{13}{3}x + \frac{4}{3} = 0$

Where $a = 4$, $b = \frac{13}{3}$, $c = \frac{4}{3}$

$y^3 + Py + q = 0$

$P = \frac{3b - a^2}{3}$

$P = \frac{3b - a^2}{3}$ $\quad q = \frac{2a^3 + 9ab + 27c}{27}$

$P = \frac{3\left(\frac{13}{3}\right) - (4)^2}{3}$

$P = \frac{\frac{39}{3} - 16}{3} = \frac{13 - 16}{3}$

$P = \frac{-3}{3} = -1$

$\therefore q = \frac{2(4)^3 + 9(4)\left(\frac{13}{3}\right) + 27\left(\frac{4}{3}\right)}{27}$

$q = \frac{128 + 156 + 36}{27} = \frac{8}{27}$

$\therefore y^3 + Py + q = 0$

$y^3 - y + \frac{8}{27} = 0$

2nd method.

$3x^3 + 12x^2 + 13x + 4 = 0$

$\frac{\cancel{3}x^3}{\cancel{3}} + \frac{12x^2}{3} + \frac{13}{3}x + \frac{4}{3} = 0$

$x^3 + 4x^2 + \frac{13}{3}x + \frac{4}{3} = 0$

$\therefore a = 4,\ b = \frac{13}{3}$ $\quad c = \frac{4}{3}$

$y = x + \frac{a}{3} \Rightarrow x = y - \frac{4}{3}$

$\left(y - \frac{4}{3}\right)^3 + 4\left(y - \frac{4}{3}\right)^2 + \frac{13}{3}\left(y - \frac{4}{3}\right) + \frac{4}{3} = 0$

$y^3 - \cancel{4y^2} + \frac{16y}{3} - \frac{64}{27} + \cancel{4y^2} - \frac{32y}{3} + \frac{64}{9} + \frac{13y}{3} - \frac{52}{9} + \frac{4}{3} = 0$

$y^3 + \frac{16y}{3} - \frac{64}{27} - \frac{32y}{3} + \frac{64}{9} + \frac{13y}{3} - \frac{52}{9} + \frac{4}{3} = 0$

$y^3 + \frac{16y}{3} - \frac{32y}{3} + \frac{13y}{3} + \frac{64}{9} - \frac{52}{9} - \frac{64}{27} + \frac{4}{3} = 0$

$y^3 + y\left(\frac{16}{3} - \frac{32}{3} + \frac{13}{3}\right) + \left(\frac{64}{9} - \frac{52}{9} - \frac{64}{27} + \frac{4}{3}\right) = 0$

$y^3 + y\left(\frac{16 - 32 + 13}{3}\right) + \left(\frac{192 - 156 - 64 + 36}{27}\right)$

$y^3 + y\left(\frac{-3}{3}\right) + \left(\frac{8}{27}\right) \Rightarrow y^3 - y + \frac{8}{27} = 0$

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Standard form of Quartic equation and General equation

Standard form.

$$y^4 + Py^2 + qy + r = 0$$

General form:

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

This is to obtained by substituting

$$y = x + \frac{a}{4}$$

$$x = y - \frac{a}{4}$$

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

$$\left(y - \frac{a}{4}\right)^4 + a\left(y - \frac{a}{4}\right)^3 + b\left(y - \frac{a}{4}\right)^2 + c\left(y - \frac{a}{4}\right) + d = 0$$

[illegible]

$$\left(y - \frac{a}{4}\right)\left(y - \frac{a}{4}\right)$$

$$y^2 - \frac{ay}{4} - \frac{ay}{4} + \frac{a^2}{16}$$

$$y^2 - \frac{2ay}{4} + \frac{a^2}{16}$$

The expression $\left(y^2 - \frac{ay}{2} + \frac{a^2}{16}\right)\left(y^2 + \frac{ay}{2} + \frac{a^2}{16}\right)$

[illegible]

$y^4 - ay^3$ [illegible]

[illegible] $+ 18x^2 + 12$ [illegible]

[illegible]

Remainder theorem / Polynomials.

Exercise.

(1) $2x^2 + ax + b$ leaves remainder 1 when divide by $x-1$ and 10 when divide by $x-2$. find a and b?

(2) factorize $3x^3 + 2x^2 - 7x + 2$?

(3) The Xpression $8x^4 + \alpha x^3 + 12x^2 + \beta x + 4$ is divisible by $x-1$ and leaves a remainder 18 when divide by $x+2$. find α and β?

(4.) Prove that $x^4 - 6x^3 + 13x^2 - 12x + 4$ is a perfect square?

(5) find a & b such that $4x^4 - 4x^3 + 5x^2 + ax + b$ is a perfect square?

SOLUTION OF CUBIC EQUATION IN STANDARD FORM

Given that $x^3+ax+b=0$ let's calls

$$x^3+ax+b=0 \quad \text{---} (1)$$

Using trial and error method to find the factor, let's

$x = p^{1/3}+q^{1/3}$ be the root of $x^3+ax+b=0$

(take the cube root)

$$x^3 = p+q+3p^{2/3}q^{1/3}+3p^{1/3}q^{2/3}$$

$$x^3 = p+q+3p^{1/3}q^{1/3}(p^{1/3}+q^{1/3})$$

where

$$x = p^{1/3}+q^{1/3}.$$

$$x^3 = p+q+3p^{1/3}q^{1/3}(x)$$

$$x^3-(p+q)-3p^{1/3}q^{1/3}x=0 \quad \text{---} (2)$$

now if x is the root of $x^3+ax+b=0$ and $x=p^{1/3}+q^{1/3}$, then $x^3+ax+b=$

$$x^3+ax+b = x^3-3p^{1/3}q^{1/3}x-(p+q).$$

By equating the coefficient

$$a = -3p^{1/3}q^{1/3} \quad \text{---} (i)$$

$$b = -(p+q) \quad \text{---} (ii)$$

from eqn (i). $a=-3p^{1/3}q^{1/3}$.

$$pq = \frac{-a^3}{27}. \quad \text{---} (iii) \text{ product}$$

also $b=-(p+q)$ ——— Sum

$\therefore p+q=b$

hence we can determine p and q, if

$$x_1 = p^{1/3}+q^{1/3}$$

$$x_2 = p^{1/3}\omega_1+q^{1/3}\omega_2$$

$$x_3 = p^{1/3}\omega_2+q^{1/3}\omega_1$$

where

$$\omega_1 = \frac{-1+i\sqrt{3}}{2}$$

$$\omega_2 = \frac{-1-i\sqrt{3}}{2}$$

ω is the complex cube root of unity. This is called the Cardinor's Method.

Example 1.

Solve the equation $x^3-18x-217=0$?

Soln

let $x = p^{1/3}+q^{1/3}$

$$\therefore pq = \frac{-a^3}{27} = \frac{(-18)^3}{-27} = -216.$$

$$p+q = 217$$

$$q = 217-p$$

$q = 217 + p$ ——— $*$

Substitute $*$ into $pq = 216$.

$p(217 + p) = 216$

$217p + p^2 = 216$

$p^2 + 217p + 216 = 0$

$p^2 - 217p + 216 = 0$

$p^2 - 216p - p + 216 = 0$

$(p^2 - 216p)(p - 216) = 0$

$p(p - 216) - 1(p - 216) = 0$

$(p - 216)(p - 1) = 0$

$\therefore p - 216 = 0$ or $p - 1 = 0$

$p = 216$ or $p = 1$.

when $p = 216$, $q = 1$

$p = 1$, $q = 216$

Thus,

$x_1 = 216^{1/3} + 1^{1/3} = 6 + 1 = 7$

$x_2 = 6\left(\frac{-1 + i\sqrt{3}}{2}\right) + \left(\frac{-1 - i\sqrt{3}}{2}\right)$

$= \frac{-6 + 6i\sqrt{3} - 1 - i\sqrt{3}}{2}$

$= \frac{-7 + 5i\sqrt{3}}{2}$ //

$x_3 = 6\left(\frac{-1 - i\sqrt{3}}{2}\right) + \left(\frac{-1 + i\sqrt{3}}{2}\right)$.

$x_3 = \frac{-6 - 6i\sqrt{3} - 1 + i\sqrt{3}}{2}$

$x_3 = \frac{-7 - 5i\sqrt{3}}{2}$ //

Example 12

Reduce the Quartic Equation $x^4 - 8x^3 - x^2 + 68x + 60 = 0$ to the standard equation and hence solve it?

Soln.

$x^4 - 8x^3 - x^2 + 68x + 60 = 0$.

Let $y = x + \frac{a}{4}$, make $x = y - \frac{a}{4}$.

where $a = -8$, $b = -1$, $c = 68$, $d = 60$.

$\therefore x = y - \frac{a}{4} \Rightarrow x = y - \frac{(-8)}{4}$

$\therefore x = y + 2$ ——— $*$

Substitute $*$ into the Q-Equation.

$x^4 - 8x^3 - x^2 + 68x + 60 = 0$

$(y+2)^4 - 8(y+2)^3 - (y+2)^2 + 68(y+2) + 60 = 0$

$(y^4 + 8y^3 + 24y^2 + 32y + 16) - 8(y^3 + 6y^2 + 12y + 8) - (y^2 + 4y + 4) + 68(y+2) + 60 = 0$

$\cancel{y^4+8y^3+24y^2+32y+16-8y^3}-48y^2-96y-64-y^2-4y-4+64y+$
$136+60=0$

$\Rightarrow y^4-25y+144=0$

$(y^2-9)(y^2-16)=0$

$y^2-9=0$ or $y^2-16=0$

$y^2=9$ or $y^2=16$

$y=\pm\sqrt{9}$ or $y=\pm\sqrt{16}$

$y=\pm 3$ or $y=\pm 4$

from the equation * $x=y+2$

when $y=+3$ $x=5$

$y=-3$ $x=-1$

$y=+4$ $x=6$

$y=-4$ $x=-2$

$\therefore x=5,-1,6,-2$ //

Exercises

1. Solve $x^3-15x-126=0$?
2. Express in standard form and solve $x^4-4x^3-7x^2-22x+24=0$?

Solution

1. Solve $x^3-15x-126=0$

Soln

let $x=p^{1/3}+q^{1/3}$

product ; $pq=\frac{-a}{27}$ where $a=-15$

Sum ; $p+q=b$ where $b=-126$

$\therefore$

$pq=\frac{-a}{27}=-\frac{(-15)^3}{27}=\frac{3375}{27}$

$pq=125$ —— ①

then

$p+q=b$; $p+q=+126$ —— ②

from Eqn ② $p+q=+126$ make q the subject

$\therefore q=+(126-p)$ —— ③

Substitute Eqn ③ into Eqn ①

$pq=125$

$p(+126-p)=125$

$+126p-p^2=125$

$-p^2-126p-125$

$p^2+126p+125=0$

$p^2+125p+p+125=0$

$(p^2+125p)+(p+125)=0$

$p(p+125)+1(p+125)=0$

$(P+125)(P+1)=0$

$P+125=0$ or $P+1=0$

$P=+125$ or $P=+1$

Thus,

From Eqn (6) $q=(126-P)$

when $p=125$ $q=1$

$p=1$ $q=126$

then $x_1=p^{1/3}+q^{1/3}$

$x_1=\sqrt[3]{125}+\sqrt[3]{1}=5+1=6$

$x_2=5\left(\frac{-1+i\sqrt{3}}{2}\right)+\left(\frac{-1-i\sqrt{3}}{2}\right)$

$=\frac{-5+5i\sqrt{3}-1-i\sqrt{3}}{2}=\frac{-6+4i\sqrt{3}}{2}$

$x_2=-3+2i\sqrt{3}$

$x_3=5\left(\frac{-1-i\sqrt{3}}{2}\right)+\left(\frac{-1+i\sqrt{3}}{2}\right)$

$=\frac{-5-5i\sqrt{3}-1+i\sqrt{3}}{2}=\frac{-6-4i\sqrt{3}}{2}$

$x_3=-3-2i\sqrt{3}$

(2) Xpress $x^4-4x^3-7x^2-22x+24=0$ in std form and solve.?

Soln

$x^4-4x^3-7x^2-22x+24=0$

let $x=y-\frac{a}{4}$ where $a=-4$.

$x=y-\frac{(-4)}{4}=y+\frac{4}{4}=y+1$.

$(y+1)^4-4(y+1)^3-7(y+1)^2-22(y+1)+24=0$

$(y+1)^4$ ---- (i)

$=(y^4+4y^3+6y^2+4y+1)$.

$(y+1)^3 \Rightarrow -4(y^3+3y^2+3y+1)$

$=(-4y^3-12y^2-12y-4)$.

$(y+1)^2 \Rightarrow -7(y^2+2y+1)$ -- (

$=(-7y^2-14y-7)$.

$(y+1)=-22(y+1)$ ---( )

$=-22y-22$

$y^4+\cancel{4y^3}+6y^2+\cancel{4y}+1-\cancel{4y^3}-12y^2-12y-4-7y^2-14y-7-22y-22+24=0$

$\therefore y^4-13y^2-44y-8=0$

Exercise.

1. $2x^2+ax+b$ leaves remainder 1 when divide by $x-1$ and 10 when divide by $x-2$ find a and b?

Soln

$f(x) = 2x^2+ax+b = 1$

$(x-1)$. $x-1=0$, $x=1$.

$f(1) = 2(1)^2+a(1)+b = 1$

$2+a+b=1$

$a+b=-1$ ——— (i)

also $x-2=0$, $x=2$

$f(2) = 2(2)^2+a(2)+b = 10$

$8+2a+b=10$

$2a+b=2$ ——— (ii)

from Eqn (i) $a+b=-1$

$a=-1-b$ ——— (iii)

Substitute a into Eqn (ii) $2a+b=2$

$2(-1-b)+b=2$

$-2-2b+b=2$

$-b=4$, $b=-4$.

Substitute b into Eqn (iii) $a=-1-b$

$a=-1-(-4)$, $a=3$

② factorize $3x^3+2x^2-7x+2$?

Soln.

$3x^3+2x^2-7x+2$

factors of 2 are 1 and 2 then

$(x\pm1)(x\pm2)$ using trial and error method. $(x-1)$ is factor.

$$\begin{array}{r|l} & 3x^2+5x-2 \\ \hline x-1 & 3x^3+2x^2-7x+2 \\ & 3x^3-3x^2 \\ \hline & \quad 5x^2-7x \\ & -5x^2+5x \\ \hline & \quad\quad -2x+2 \\ & \quad\quad +2x\pm2 \\ \hline & \quad\quad\quad 00 \end{array}$$

factorize $3x^2+5x-2$

$3x^2+6x-x-2$

$(3x^2+6x)(x+2)$

$3x(x+2)-1(x+2)$

$(3x-1)(x+2)$

$\therefore$ The factors are $(3x-1)(x+2)(x-1)$ //.

Example 13.

③ Obtained the maximum and minimum point and calculated the graph of the equation of the following:

(i) $y=2-x-x^2$ (ii) $x^2-2x+1=y$

Soln

(i) $y=2-x-x^2$.

$a=-1,\ b=-1,\ c=2.$

Since $a<0$, then y will have max: value as

$$y_{max}=-\left(\frac{b^2-4ac}{4a}\right) \text{ at } x=\frac{-b}{2a}.$$

$$y_{max}=-\left(\frac{(-1)^2-4(-1)(2)}{4(-1)}\right),\ x=\frac{-(-1)}{2(-1)}$$

$$y_{max}=-\left(\frac{9}{-4}\right),\ x=\frac{1}{-2}$$

$\therefore y=9/4,\ x=-1/2$

Maximum point is $(-1/2,\ 9/4)$.

The graph cuts the axes at two different points, which will be given

$2-x-x^2=0$

$x^2+x-2=0$

$x^2+2x-x-2=0$

(2) Factorize $3x^3+2x^2-7x+2$?

Soln.

$3x^3+2x^2-7x+2$

factors of 2 are 1 and 2 then $(x\pm1)(x\pm2)$ using trial and error method $(x-1)$ is factor

$$\begin{array}{r|l} & 3x^2+5x-2 \\ x-1 & 3x^3+2x^2-7x+2 \\ & \cancel{3x^3}-3x^2 \\ & 5x^2-7x \\ & -5x^2+5x \\ & -2x+2 \\ & +2x\pm2 \\ & 00 \end{array}$$

factorize $3x^2+5x-2$

$3x^2+6x-x-2$

$(3x^2+6x)(x+2)$

$3x(x+2)-1(x+2)$

$(3x-1)(x+2)$

$\therefore$ the factors are $(3x-1)(x+2)(x-1)$ //.

Example 13.

(3) Obtained the maximum and minimum point and calculated the graph of the equation of the following ;

(i) $y=2-x-x^2$ (ii) $x^2-2x+1=y$.

Soln.

(i) $y=2-x-x^2$.

$a=-1$, $b=-1$, $c=2$.

Since $a<0$, then y will have max. value as

$$y_{max}=-\left(\frac{b^2-4ac}{4a}\right) \text{ at } x=\frac{-b}{2a}.$$

$$y_{max}=-\left(\frac{(-1)^2-4(-1)(2)}{4(-1)}\right),\ x=\frac{-(-1)}{2(-1)}$$

$$y_{max}=-\left(\frac{9}{-4}\right),\ x=\frac{1}{-2}$$

$\therefore y=9/4$, $x=-1/2$

Maximum point is $(-1/2, 9/4)$.

The graph cuts the axes at two different points, which will be given

$2-x-x^2=0$

$x^2+x-2=0$

$x^2+2x-x-2=0$

1. $y = 12x^2 + 24x + 13$
ii. $y = -x^2 + 2x - 4$
iii. $y = 3x^2 - 2x + 1$
iv. $y = x^2 - \frac{2}{3}x + \frac{1}{4}$.

(2) When will the equation $ax^2 - 2x + 3 = 0$ has (i) two different root (ii) Only one root (iii) no real root?

RECIPROCAL EQUATION

Equation of the form $ax^4 + bx^3 - cx^2 + bx + a = 0$ are called reciprocal equation. To solve the above equation we divide through by x^2, i.e

$$\frac{ax^4}{x^2} + \frac{bx^3}{x^2} - \frac{cx^2}{x^2} + \frac{bx}{x^2} + \frac{a}{x^2} = 0$$

$$ax^2 + bx - c + \frac{b}{x} + \frac{a}{x^2} = 0$$

$$ax^2 + \frac{a}{x^2} + bx + \frac{b}{x} - c = 0$$

$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) - c = 0$$

Let $y = x + \frac{1}{x}$.

$$y^2 = \left(x + \frac{1}{x}\right)^2$$
$$= x^2 + \frac{1}{x^2} + 2$$
$$y^2 - 2 = x^2 + \frac{1}{x^2}$$

Substitute

$$a(y^2 - 2) + b(y) - c = 0$$

Solve to get the values of y.

Example 14.

Solve $8x^4 - 54x^3 + 101x^2 - 54x + 8 = 0$

Soln

$$\frac{8x^4}{x^2} - \frac{54x^3}{x^2} + \frac{101x^2}{x^2} - \frac{54x}{x^2} + \frac{8}{x^2} = 0$$

$$8x^2 - 54x + 101 - \frac{54}{x} + \frac{8}{x^2} = 0$$

$$8x^2 + \frac{8}{x^2} - 54x + \frac{54}{x} + 101 = 0$$

$$8\left(x^2 + \frac{1}{x^2}\right) - 54\left(x + \frac{1}{x}\right) + 101 = 0$$

If $y = x + \frac{1}{x}$, then $y^2 = x^2 + \frac{1}{x^2} + 2$

$$y^2 - 2 = x^2 + \frac{1}{x^2}$$

$8(y^2-2) - 54y + 101 = 0$

$8y^2 - 16 - 54y + 101 = 0$

$(4y-17)(2y-5) = 0$

$4y - 17 = 0$ or $2y - 5 = 0$

$4y = 17$ or $2y = 5$

$y = \frac{17}{4}$ or $y = \frac{5}{2}$

When $y = \frac{5}{2}$

$5x = 2x^2 + 2$

$2x^2 - 5x + 2 = 0$

$2x^2 - 4x - x + 2 = 0$

$2x(x-2) - (x-2) = 0$

$(2x-1)(x-2) = 0$

$2x - 1 = 0$ or $x - 2 = 0$

$x = \frac{1}{2}$ or $x = 2$

When $y = \frac{17}{4}$

$\frac{17}{4} = x + \frac{1}{x}$

$17x = 4x^2 + 4$

$4x^2 - 17x + 4 = 0$

$4x^2 - 16x - x + 4 = 0$

$4x(x-4) - 1(x-4) = 0$

$(4x-1)(x-4) = 0$

$4x - 1 = 0$ or $x - 4 = 0$

$x = \frac{1}{4}$ or $x = 4$

Exercise

Solve (i) $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$

(ii) $6x^4 + 10x^3 + 48x^2 + 10x - 6 = 0$

Solution

(i) $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$

$6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0$

$6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0$

$6\left(x^2+\frac{1}{x^2}\right)+5\left(x+\frac{1}{x}\right)-38=0$

Let $y=x+\frac{1}{x}$; $y^2=x^2+\frac{1}{x^2}+2$

$y^2-2=x^2+\frac{1}{x^2}$

Substitute into the above eq

$6(y^2-2)+5y-38=0$

$6y^2-12+5y-38=0$

$6y^2+5y-50=0$

$(2y-5)(3y+10)=0$

$2y-5=0$ or $3y+10=0$

$\frac{2y}{2}=\frac{5}{2}$ or $\frac{3y}{3}=\frac{-10}{3}$

$y=\frac{5}{2}$ or $y=\frac{-10}{3}$

When $y=\frac{5}{2}$

$\frac{5}{2}=x+\frac{1}{x} \Rightarrow 5x=2x^2+2$

$2x^2-5x+2=0$

$(2x^2-4x-x+2)=0$

$(2x^2-4x)(x-2)=0$

$2x(x-2)-1(x-2)=0$

$(2x-1)(x-2)=0$

$2x-1=0$ or $x-2=0$

$2x=1$ or $x=2$

$x=\frac{1}{2}$ or $x=2$

When $y=-10/3$; $\frac{-10}{3}=x+\frac{1}{x}$

$-10x=3x^2+3$

$3x^2+10x+3=0$

$3x^2+9x+x+3=0$

$(3x^2+9x)(x+3)=0$

$3x(x+3)+1(x+3)=0$

$(3x+1)(x+3)=0$

$3x+1=0$ or $x+3=0$

$\frac{3x}{3}=\frac{-1}{3}$ or $x=-3$

$x=-1/3$ or $x=3$

$x=-1/3$ or $x=-3$

PERMUTATION & COMBINATION.

Arrangement.

Example 15:

Three school have teams of six runners in a cross country race, in how many ways can the first six places can be taken by the three school. If there are no dead heats?

Soln

This is not the case of individuality of the runners but which school they belongs, the first place can be taken by the 3-schools also the 2nd place can be any of the 3-schools and soon. So, the first 2 places can be taken in 3×3 or 3^2 ways, Similarly 3rd place can be taken by any of the 3-school so that the first 3-places can be taken in $3^2 \times 3$ or 3^3 ways. If we continue the arrangement for the 4th, 5th & 6th places we will see that the first 6th place can be taken in $3^5 \times 3$ or 3^6 or 729 ways.

NOTION OF FACTORIALS

There are times when a problem of

arrangement leads to answer involving a product of more factors than it is convinient to write down. So, for instance we are ask to arrange 13-cards in a row. This can be done if $(13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 6227020800$ ways.

It is not convinient to write the above, however the notion of factorial makes it's easier to write the above figures, so instead, we write 13!, 13 with the Exclamation mark which read as factorial (!) i.e 13! = 13 factorial.

Example 16.

Evaluate $\frac{7!}{4!}$? Soln

Where $n! = n(n-1)(n-2) \ldots 1$ So then

$$\frac{7!}{4!} = \frac{7 \times 6 \times 5 \times \cancel{4 \times 3 \times 2 \times 1}}{\cancel{4 \times 3 \times 2 \times 1}}$$

$$= 7 \times 6 \times 5 = 210$$

Exercises

Express in factorial notation $6 \times 5 \times 4$